Milewski

Berlin

L Milewski

DE ABELIANARUM FUNCTIONUM PERIODIS
PER AEQUATIONES DIFFERENTIALES DEFINIENDIS.

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AUCTOR.

De abelianarum functionum periodis per aequationes differentiales definiendis.

Constat periodos functionum abelianarum integralia esse „confecta“,[1]) quae vocantur. Sed lineae, secundum quas fit integratio[2]) raro tantum, ubi minores se difficultates praebent, re vera inveniri possunt. Quum vero in expressionibus finalibus periodûm, linearum istarum nullum vestigium appareat, in rem fore visum est, perquirere, utrum sint methodi periodorum definiendarum, quae his ex lineis non pendeant. Hujusmodi methodum Vir Illustrissimus Weierstrass in medium protulit atque aestate anni 1874 privatissime paucis in lectionibus de functionibus hyperellipticis disserens cum audientibus communicavit. Est methodus haec, periodorum per aequationes differentiales lineares definiendarum.

Sumitur hac in methodo, datos esse valores periodûm pro aliquo modulorum systemate. Si igitur $\rho=3$ ponatur, ubi ρ ordinem[3]) aequationis designat, ejusmodi aequationes constitui possunt, ubi integralia respectiva primae speciei ad elliptica revocentur, unde fit, ut calculari possint. Jam et pro majoribus ρ similes aequationes inveneris, neque dubito, quin pro omni ρ aequationes hujusmodi statui possint. Quae quum ita sint, omnis aequatio uniuscujusque ordinis ad aequationem hujusmodi per variationem constantium revocabitur, et aequationes differentiales, quibus periodi definiantur, generaliter solventur. Tali quoque modo vera analytica indoles atque natura periodorum nostrarum optime elucebit.

Itaque operae pretium duxi, methodum hanc ad integralia hyperelliptica adhibere, quamquam jam constat, periodos ipsorum analyticas esse functiones modulorum. Simili modo in abelianis integralibus et ipsis, ubi $\rho=3$ est, aequationes illas differentiales formaturus sum. Quod problema eo perducere mihi contigit, ut solutio finalis ipsius non

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[1]) Vollständige oder geschlossene Integr.

[2]) Die Integrationswege.

[3]) Der Rang.

pendeat, nisi ab aequationum linearium systemate, cujus determinans non evanescit, resolvendo. Valores autem numerici calculari mihi non posse videntur, quoad moduli aequationis, a qua proficisci nos oporteat, plane generales maneant.

I.

Integralia hyperelliptica.

Proficiscamur, Ill. Weierstrass sequentes, ab illa aequatione:

1) $$y^2 = 4x^{2\rho+1} + g_1 x^{2\rho} + g_2 x^{2\rho-1} + \dots + g_{2\rho} x + g_{2\rho+1}$$

$$\text{vel } y^2 = R(x) = \sum_{0}^{2\rho+1}{}_{\lambda}\, g_{2\rho+1-\lambda} x^{\lambda} \qquad (g_0 = 4)$$

ubi modulus g_1 semper nullitati aequalis poni, et etiam aliquis alius modulus ad libitum sumi potest. Liceat autem symmetriae causa hac generali forma uti. Pro magnis valoribus variabilis x hoc modo repraesentatur y:

$$\bar{y}_t = 2\mathfrak{t}^{-2\rho-1} \sum_{0}^{\infty}{}_{\alpha}\, \mathfrak{a}_{\alpha} \mathfrak{t}^{2\alpha}, \text{ si } \bar{x}_t = \mathfrak{t}^{-2}$$

$$\text{ubi } \mathfrak{a}_1 = \frac{g_1}{8}, \mathfrak{a}_2 = \frac{16 g_2 - g_1^2}{2.\ 8.\ 8.}, \mathfrak{a}_3 = \frac{128 g_3 - 16 g_1 g_2 + g_1^3}{2.\ 8^3},$$

$$\mathfrak{a}_4 = \frac{4096 g_4 - 256 g_2^2 + 512 g_1 g_3 - 32 g_1^2 g_2 + 3 g_1^4}{8^5}$$

etc.

Functiones $H(xy)_\alpha$ et $H'(xy)_\alpha$, quae integralia resp. primae ac secundae speciei nobis suppeditant, definiuntur, si significationibus notis utimur, hoc modo

$$\underset{k\alpha}{H}(x'y')_\alpha = \left[H(\bar{x}_t \bar{y}_t, x'y') \right]_{t-k\alpha} \qquad (\alpha = 1, 2, \dots \rho)$$

$$H(x'y') = \left[\underset{k\alpha}{H}(\bar{x}_t \bar{y}_t, x'y')) \right]_{\mathfrak{t}^{k\alpha}}$$

Tum erit $H'(x'y')_\alpha = -k_\alpha \underset{k\alpha}{H}(x'y')$. Est autem $H(xyx'y') = \frac{1}{2} \frac{y'+y}{x'-x} \frac{1}{y'}$

$$k_1 = 2\rho - 1,\ k_2 = 2\rho - 3, \dots k_\rho = 1$$

vel generaliter $k_\alpha = 2\rho + 1 - 2\alpha$

Hoc modo invenimus:

$$H(x'y')_\alpha = -\frac{\mathfrak{a}_{\alpha-1} + \mathfrak{a}_{\alpha-2}\,x'^2 + \ldots + \mathfrak{a}_1\,x'^{\alpha-2} + x'^{\alpha-1}}{y'}$$

$$\overset{k\alpha}{H}(x'y') = -\frac{\mathfrak{a}_{2\rho-\alpha} + \mathfrak{a}_{2\rho-\alpha-1}\,x' + \mathfrak{a}_{2\rho-\alpha-2}\,x'^2 + \ldots + \mathfrak{a}_1\,x'^{2\rho-\alpha-2} + x'^{2\rho-\alpha}}{y}$$

ergo:

$$H(xy)_1 = -\frac{1}{y}$$

$$H(xy)_2 = -\frac{\mathfrak{a}_1 + x}{y}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$H(xy)_\rho = -\frac{\mathfrak{a}_{\rho-1} + \mathfrak{a}_{\rho-2}\,x + \ldots + \mathfrak{a}_1\,x^{\rho-2} + x^{\rho-1}}{y}$$

$$H'(xy)_1 = (2\rho-1)\,\frac{\mathfrak{a}_{2\rho-1} + \mathfrak{a}_{2\rho-2}\,x + \ldots + \mathfrak{a}_2\,x^{2\rho-3} + \mathfrak{a}_1\,x^{2\rho-2} + x^{2\rho-1}}{y}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$H'(xy)_2 = (2\rho-3)\,\frac{\mathfrak{a}_{2\rho-2} + \mathfrak{a}_{2\rho-1}\,x + \ldots + \mathfrak{a}_2\,x^{2\rho-4} + \mathfrak{a}_1\,x^{2\rho-2} + x^{2\rho-2}}{y}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$H'(xy)_{\rho-1} = 3\,\frac{\mathfrak{a}_{\rho+1} + \mathfrak{a}_\rho\,x + \ldots + \mathfrak{a}_1\,x^{\rho} + x^{\rho+1}}{y}$$

$$H'(xy)_\rho = \frac{\mathfrak{a}_\rho + \mathfrak{a}_{\rho-1}\,x + \ldots + \mathfrak{a}_1\,x^{\rho-1} + x^{\rho}}{y}$$

Quum functionum $H(xy)_\alpha$ loco, ρ lineares functiones ipsarum, inter quas nulla sit relatio linearis, poni possint, his simplicissimis utamur formis:

$$\overline{H}(xy)_1 = \frac{1}{y}, \quad \overline{H}(xy)_2 = \frac{x}{y}, \ldots \quad \overline{H}(xy)_\alpha = \frac{x^{\alpha-1}}{y}, \ldots \quad \overline{H}(xy)_\rho = \frac{x^{\rho-1}}{y}$$

Functiones correspondentes $\overline{H}'(xy)_\alpha$ facile definiuntur per illam aequationem, maximi in hac theoria momenti:

$$2)\quad \frac{d}{dx}\,H(xyx'y') - \frac{d}{dx'}\,H(x'y'xy) = \sum_{1}^{\rho}{}_{\alpha} \left\{ H(x'y')_\alpha\, H'(xy)_\alpha - H(xy)_\alpha\, H'(xy')_\alpha \right\},$$

quae per mutationem nostram turbari non debet. Est autem:

$$\frac{d}{dx} H(xyx'y') - \frac{d}{dx} H(x'y'xy) = \frac{d}{dx} \frac{y+y'}{2(x'-x)y'} - \frac{d}{dx'} \frac{y'+y}{2(x-x')y}$$

$$= \frac{d}{dx} \frac{y}{2(x'-x)y'} - \frac{d}{dx'} \frac{y'}{2(x-x')y}$$

$$= \frac{1}{4yy} \left\{ \frac{d}{dx'} \frac{R(x)-R(x')}{x-x'} - \frac{d}{dx} \frac{R(x)-R(x')}{x-x'} \right\}$$

Tum est

$$\frac{1}{4yy'} \left\{ \frac{d}{dx'} \frac{Rx-Rx'}{x-x'} - \frac{d}{dx} \frac{Rx-Rx'}{x-x'} \right\} = \frac{1}{4yy'} \sum_2^{2\rho}{}_\lambda\, g_{2\rho-\lambda} \sum_{\alpha,\beta} (\beta-\alpha)\, x^\alpha x'^\beta$$

$$(\beta+\alpha=\lambda-1)$$

Sit λ' maximus numerus integer $\leqq \frac{\lambda-1}{2}$. E numero coëfficientium $(\beta-\alpha)$ tot esse positivos, quot negativos apparet. Priores hoc modo scribi possunt:

$$(\lambda-1-2\alpha) \text{ ubi } \alpha=0,1,2,..\lambda',$$

alteri coëfficientes simili modo erunt:

$$-(\lambda-1-2\beta) \text{ ubi } \beta=0,1,..\lambda'$$

$$\text{Ponatur: } (x,x')_{\lambda-1} = \sum_o^{\lambda'}{}_\beta\, (\lambda-1-2\beta)\, x^{\lambda-1-\beta}\, x'^\beta$$

tum erit:

$$(x,x')_1 = x$$
$$(x,x')_2 = 2x^2$$
$$(x,x')_3 = 3x^3 + x^2 x'$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$(x,x')_{2\rho-1} = (2\rho-1)\, x^{2\rho-1} + (2\rho-3)\, x^{2\rho-2} x' + .. + x^\rho x'^{\rho-1}$$

$$\frac{1}{4yy'} \left\{ \frac{d}{dx'} \frac{Rx-Rx'}{x-x'} - \frac{d}{dx} \frac{Rx-Rx'}{x-x'} \right\} = \frac{1}{4yy'} \sum_2^{2\zeta}{}_\lambda\, g_{2\rho-\lambda} \left\{ (x',x)_{\lambda-1} - (x,x')_{\lambda-1} \right\}$$

$$= \frac{1}{yy'} \left\{ \frac{4(x'x)_{2\rho-1} + g_1 (x'x)_{2\rho-2} + \ldots + g_{2\rho-2}(x'x)_1}{4} - \frac{4(xx')_{2\rho-1} + g_1 (xx')_{2\rho-1} + \ldots + g_{2\rho-2}(xx')_1}{4} \right\}$$

$$= \frac{H(xx') - H(x'x)}{yy'}$$

si $H(xx') = -\frac{1}{4}\left(4(xx')_{2\rho-2} + g_1 (xx')_{2\rho-2} + \cdot\cdot g_{2\rho-2}(x'x)_1 \right)$

$$= \frac{1}{4} \sum_2^{2\rho}{}_\lambda\, g_{2\rho-\lambda}\, (xx')_{\lambda-1} \text{ ponitur.}$$

Functioni $H(x\,x')$ etiam haec tribui potest forma,

$$H(x\,x') = H(x)_1 + H(x)_2\, x' + \ldots + H(x)_\rho\, x'^{\rho-1}$$

ubi $H(x)_1 = -\frac{1}{4}\left(\overline{2\rho-1}\, 4\,x^{2\rho-1} + \overline{2\rho-2}\, g_1\, x^{2\rho-2} + \ldots 3\, g_{2\rho-4}\, x^3 + 2 g_{2\rho-3}\, x^2 + g_{2\rho-2}\, x\right)$

$$H(x)_2 = -\frac{1}{4}\left(\overline{2\rho-3}\; 4x^{2\rho-2} + \overline{2\rho-4}\; g_1\, x^{2\rho-3} + \ldots + g_{2\rho-4}\, x^2\right)$$

. . . .

$$H(x)_\rho = -\frac{1}{4}\cdot 4\, x^\rho = -x^\rho$$

Hanc propter hoc habemus aequationem:

3) $\dfrac{d}{dx} H(x\,y\,x'\,y') - \dfrac{d}{dx'} H(x'\,y'\,x\,y) =$

$$\frac{1}{y\,y'}\left\{H(x)_1 + H(x)_2\, x' + \ldots + H(x)_\rho\, x'^{\rho-1} - H(x')_1 - H(x')_2\, x - \ldots - H(x')_\rho\, x^{\rho-1}\right\}$$

Quum autem esse debeat:

$$\frac{d}{dx} H(x\,y\,x'\,y') - \frac{d}{dx'} H(x'\,y'\,x\,y) = \sum_1^\rho{}_\alpha \left(H(x'\,y')_\alpha\, H'(x\,y)_\alpha - H(x\,y)_\alpha\, H'(x'\,y')_\alpha\right)$$

et pro functionibus $H(x\,y)_\alpha$ positum sit resp. $\dfrac{x^{\alpha-1}}{y}$, si pro $H'(x\,y)_\alpha$ quaedam functiones $\overline{H}'(x\,y)_\alpha$ adhuc incognitae inducantur, haec apparebit relatio:

$$\frac{d}{dx} H(x\,y\,x'\,y') - \frac{d}{dx'} H(x'\,y'\,x\,y) = \sum_1^\rho \left(\frac{x'^{\alpha-1}}{y'}\overline{H}'(xy)_\alpha - \frac{x^{\alpha-1}}{y}\,\overline{H}'(x'\,y')_\alpha\right)$$

Aequatio haec comparata cum illa (3) docet, has necessario exstare aequationes

$$\overline{H}'(\alpha\,y)_\alpha = \frac{H(x)_\alpha}{y} \qquad (\alpha = 1, 2, \ldots \rho)$$

Si brevitatis causa iterum $H(x\,y)_\alpha$ et $H'(x\,y)_\alpha$ pro $\overline{H}(x\,y)_\alpha$ et $\overline{H}'(xy)_\alpha$ scribitur, apparebit

$$H(x\,y)_1 = \frac{1}{y}$$

$$H(x\,y)_2 = \frac{x}{y}$$

. . . .

$$H(x\,y)_\rho = \frac{x^{\rho-1}}{y}$$

$$H'(x\,y)_1 = -\frac{1}{4}\,\frac{\overline{2\rho-1}\cdot 4\,x^{2\rho-1} + \overline{2\rho-2}\,g_1 x^{2\rho-2} + \ldots + 2\,g_{2\rho-3}x^2 + g_{2\rho-2}\,x}{y}$$

$$= H(x\,y)_{2\rho}$$

$$H'(x\,y)_2 = -\frac{1}{4}\,\frac{\overline{2\rho-3}\;4\,x^{2\rho-2} + \overline{2\rho-4}\,g_1\,x^{2\rho-3} + \ldots + 2\,g_{2\,\rho-5}\,x^3 + g_{2\rho-4}\,x^2}{y}$$
$$= H(x\,y)_{2\rho-1}$$

. . . .

$$H'(x\,y)_{\rho-1} = -\frac{1}{4}\,\frac{3.\,4.\,x^{\rho+1} + 2\,g_1\,x^{\rho} + g_2\,x^{\rho-1}}{y} = H(x\,y)_{\rho+2}$$

$$H'(x\,y)_{\rho} = -\frac{x^{\rho}}{y} = H(x\,y)_{\rho+1}$$

Generaliter erit:

3a) $$H'(x\,y)_{\alpha} = H(x\,y)_{2\rho+1-\alpha} = -\frac{1}{4}\,\frac{(2\rho-2\alpha+1)\,4\,x^{2\rho-\alpha} + (2\rho-2\alpha)\,g_1\,x^{2\rho-\alpha-1}}{y}$$
$$\frac{+ \ldots + g_{2\rho-2\alpha}\,x^{\alpha}}{y}$$

$$= -\frac{1}{4\,y}\sum_{\alpha}^{2\rho-\alpha}{}_{\beta}\,(\beta-\alpha+1)\,g_{2\rho-\alpha-\beta}\,x^{\beta}$$
$$(\alpha = 1, 2, \ldots \rho)$$

Reciproce habemus:

$$\frac{x^{\alpha-1}}{y} = H(x\,y)_{\alpha} \qquad \alpha = 1,\,2,\ldots\rho$$

$$\frac{x^{\rho}}{y} = -H'(x\,y)_{\rho} = g'_{\rho+1,\rho}\,H'(x\,y)_{\rho}$$

Erit porro:

$$\frac{x^{\rho+\alpha}}{y} = \sum_{\rho-\alpha}^{\rho}{}_{t}\;g'_{\rho+\alpha+1,\,t}\,H'(x\,y)_t + \sum_{\rho-\alpha+1}^{\rho} g_{\rho+\alpha+1,\,t}\,H(x\,y)_t$$

sed haec conditio accedit $1 \leqq \alpha \leqq \rho - 1$.

Coefficientes $g_{m,n}$ in omni casu speciali facile determinantur.

A.

Methodus prior.

Formatis functionibus $H(x\,y)_{\alpha}$ et $H'(x\,y)_{\alpha}$ integralia quaesita primae ac secundae speciei erunt haec:

$$\int H(x\,y)_{\alpha}\,d\,x \text{ et } \int H'(x\,y)_{\alpha}\,d\,x$$

Sint $I_{\alpha} = \overline{\int H(x\,y)_{\alpha}}\,d\,x$, $I'_{\alpha} = \overline{\int H'(x\,y)_{\alpha}}\,d\,x$ quaedam periodi eorum, id est, integralia

„confecta", quorum integrationis linea (Integrationsweg) parem numerum punctorum e_α amplectatur, in quibus discrimans Rx aequationis nostrae $y^2 = Rx$, evanescat. Mutatis paullulum modulis $g_1\ g_2, \ldots g_{2\rho+1}$, et valores $e_1,\ e_2, \ldots e_{2\rho+1}$, paullulum tantum mutantur, ita ut nihil impediat, quin eisdem lineis integrationis et nunc utamur. Quum mutatae functiones I_ν lineariter per primitivas functiones exprimantur, (quod paullo post ostendetur), si modulos g functiones analyticas cujuspiam variabilis t esse fingemus, aequationes differentiales lineares primi ordinis inter $I_1\ I_2 \ldots I_\rho,\ I'_1,\ I'_2, \ldots I'_\rho$ et variabilem t habebimus. Quod ut fieri possit, haec una invenietur conditio: ne habeat $Rx = o$ radices multiplices. Cavendum igitur erit, ne modulis g valores tribuantur pro quibus discrimans aequationis $Rx = o$ evanescat.

Componuntur functiones H_α (quo signo et functiones $H(xy)_\alpha$ et, si $\alpha > \rho$ functiones H'_α amplectimur) e functionibus hujus formae $\frac{x^{\mathfrak{a}}}{y}$; integralia igitur ipsa constant ex integralibus $\int \frac{x^{\mathfrak{a}}\, dx}{y}$. De his igitur primo loco tractemus.

Formandum est $\delta \int \frac{x^{\mathfrak{a}}\, dx}{y}$; operatio haec hoc symbolo continetur $\sum \frac{d}{d g_\lambda}\, \delta g_\lambda$ quam operationem sub integrationis signo fieri posse facile intelligitur. Habemus igitur, si $\sqrt{Rx}$ pro y scribitur

$$\delta \int \frac{x^{\mathfrak{a}}\, dx}{\sqrt{Rx}} = - \int \frac{x^{\mathfrak{a}}\, \delta Rx}{2\, Rx} \frac{dx}{\sqrt{Rx}}$$

est autem

$$Rx = 4\, x^{2\rho+1} + \sum_{1}^{2\rho+1}{}_{\lambda}\, g_\lambda\, x^{2\rho+1-\lambda}$$

$$\delta Rx = \sum_{1}^{2\rho+1}{}_{\lambda}\, \delta g_\lambda\, x^{2\rho+1-\lambda}$$

4) ergo:
$$\delta \int \frac{x^{\mathfrak{a}}\, dx}{\sqrt{Rx}} = - \sum_{1}^{2\rho+1}{}_{\lambda}\, \delta g_\lambda \int \frac{x^{2\rho+\mathfrak{a}+1-\lambda}}{2\, Rx} \frac{dx}{\sqrt{Rx}}$$

Maximus valor exponentis $2\rho + \mathfrak{a} + 1 - \lambda$ est $4\rho - 1$, quum λ sit minime $= 1$ et maxime $2\rho - 1$, $\mathfrak{a}$ autem $\leqq 2\rho - 1$.

Quum variationum nostrarum δg_λ nullo modo alia ab alia pendeat, unumquodquam integrale $\int \frac{x^{2\rho+\mathfrak{a}+1-\lambda}}{2\, Rx} \frac{dx}{\sqrt{Rx}}$ per se tractari potest.

Sit $f_n(x) = \int \frac{x^n\, dx}{2\, R x \sqrt{R x}}$. Demonstrandum est, $f n(x)$ ad integralia primae ac secundae speciei revocari posse. Quod demonstratum erit, ut neminem certe fugit, si ostendetur $fn(x)$ evolutam in seriem non cohibere usquam membra logarithmica, quae vocantur, vel, quod idem est, non inveniri argumenti potestatem $(x-\alpha)^{-1}$ in ulla functionis $df_n(x)$ evolutione.

Functio $df_n(x)$ evoluta non continet omnino ullas negativas argumenti potestates nisi pro valoribus x permagnis (infinitis, ut dicitur), vel in eorum valorum e_α circuitu, pro quibus Rx evanescit. Quum constitutum sit, esse $R'(e_\gamma) \lessgtr O$, habemus:

$$Rx = R'(e_\gamma)(x-e_\gamma) + \frac{R''(e_\gamma)}{1.\ 2.}(x-e_\gamma)^2 + \ldots + \frac{R^{(2\rho+1)}(e_\gamma)}{(2\rho+1)!}(x-e_\gamma)^{2\rho+1}$$

$$= R'(e_\gamma)(x-e_\gamma)\left\{1 + \frac{R''(e_\gamma)}{2\, R'(e_\gamma)}(x-e_\gamma) + \ldots\ldots\right\}$$

$$\sqrt{Rx} = \sqrt{R'e_\gamma}\sqrt{x-e_\gamma}\left\{1 + \frac{1}{4}\frac{R''(e_\gamma)}{R'(e_\gamma)}(x-e_\gamma) + \ldots\right\}$$

$$\frac{1}{\sqrt{Rx}} = \frac{1}{\sqrt{R'e_\gamma}}\frac{1}{\sqrt{x-e_\gamma}}\left\{1 + \mathfrak{P}(x-e_\gamma)\right\}$$

$$\frac{1}{Rx} = \frac{1}{R'(e_\gamma)}\frac{1}{(x-e_\gamma)}\left\{1 + \overline{\mathfrak{P}}(x-e_\gamma)\right\}$$

$$\frac{1}{Rx\sqrt{Rx}} = \frac{1}{(x-e_\gamma)\sqrt{x-e_\gamma}\,(\sqrt{R'e_\gamma})^3}\left\{1 + \overline{\overline{\mathfrak{P}}}(x-e_\gamma)\right\}$$

Series infinita uncis inclusa positivas et integras tantum continet potestates argumenti $(x-e_\gamma)$, unde fit, ut hae duae solum reperiantur potestates negativae $(x-e_\gamma)^{-\frac{3}{2}}$, $(x-e_\gamma)^{-\frac{1}{2}}$, non reperiatur $(x-e_\gamma)^{-1}$. Quum e_γ quaeque radix sit aequationis $R(x) = O$, jam facile intelligi potest, $df_n(x)$, pro $x = \infty$ evolutam potestatem x^{-1} etiam non continere. Scimus enim summam coëfficientium primarum negativarum potestatum, quae in omnibus evolutionibus functionis $df_n(x)$ inveniantur, aequalem esse nullitati, coëfficiens igitur in serie pro $x = \infty$, aequalis est negativae summae omnium aliorum coëfficientium, qui quum omnes evanescant, et noster coëfficiens nullitati aequalis evadit. Quod quum constet, jam statim sequitur, $f_n(x)$ esse integrale primae vel secundae speciei; quam ob rem poni potest

$$5)\quad f_n(x) = \int \frac{x^n\, dx}{2\, Rx\, \sqrt{Rx}} = \sum_{\alpha=1}^{\rho} \frac{1}{2\, L}\, h_{n,\,\alpha} \int H(xy)_\alpha\, dx$$

$$+ \sum_{\alpha=1}^{\rho} \frac{1}{2\, L}\, h'_{n,\,\alpha} \int H'(xy)_\alpha\, dx + \frac{\psi_n(x\, y)}{y}$$

ubi $h_{n,\alpha}$ indefinitae adhuc functiones sunt modulorum $g_1, g_2, \dots g_{2\rho+1}$, $2L$ indicem communem designat, $\psi(x\,y)$ autem functio rationalis est variabilium x, y. Quam functionem ψ, functionem integram esse nunc demonstrabitur.

Habemus $$\frac{d f_n(x)}{dx} = \frac{x^n}{2\,R\,x\,\sqrt{R\,x}} = \sum_{\alpha=1}^{\alpha=\rho} \frac{1}{2L}\left\{h_{n,\alpha}\,H(x\,y)_\alpha + h'_{n,\alpha}\,H'(x\,y)_\alpha\right\}$$
$$+ \frac{d}{dx}\frac{\psi_n(x\,y)}{y} = \frac{\phi_n(x)}{y} + \frac{d}{dx}\frac{\psi_n(x\,y)}{y}$$

ubi $\phi_n(x) = l_{n,0} + l_{n,1}\,x + l_{n,2}\,x^2 + \dots + l_{n,2\rho-1}\,x^{2\rho-1}$ est, ψ autem hac formula continetur: $P(x) + y\,Q(x)$, ubi Px et Qx functiones rationales sunt solius x. Sequitur igitur:

$$\frac{x^n}{2y^3} = \frac{\phi x}{y} + \frac{2\,R(x)\,\left(P'(x) + y\,Q'(x)\right) + R'(x)\,\left(Q(x)\,y - P - Qy\right)}{2\,y^3}$$

quum sit $\frac{dy}{dx} = \frac{R'\,x}{2\,y}$

vel etiam

$$x^n = 2\,R(x)\,\left(\phi x + P'(x)\right) - R'(x)\;P(x) + 2\;Q'(x)\,R(x)\,\sqrt{R\,x}$$

Quum aequatio haec identica sit, coëfficientem radicis $\sqrt{R\,x}$ evanescere necesse est; est igitur $Q' = O = \frac{dQ}{dx}$, id est $Q = \text{constans}$.

Restat, ut functio $P(x)$ determinetur. Observemus aequationem:

5a) $$f_n(x) = \int \frac{x^n\,dx}{2\,Rx\,\sqrt{Rx}} = \int \frac{\phi_n(x)\,dx}{y} + \frac{P_n(x) + y\,Q_n}{y}$$

vel etiam: $$\int \frac{x^n\,dx}{2\,R\,x\,\sqrt{R\,x}} - \int \frac{\phi_n(x)\,dx}{\sqrt{R\,x}} = \frac{P_n(x) + Q_n\,y}{y}$$

Est autem

$$\frac{x^n}{2\sqrt{R\,x}\;R\,x} = \frac{(e_\gamma + x - e_\gamma)^n}{2\;Rx\;\sqrt{Rx}} = \frac{e_\gamma^n + n\,e_\gamma^{n-1}\,(x - e_\gamma) + \dots + (x - e_\gamma)^n}{2}\;\frac{\left\{1 + \bar{\bar{\mathfrak{P}}}\,(x - e_\gamma)\right\}}{(\sqrt{x - e_\gamma}\;\sqrt{R'\,e_\gamma})^3}$$

$$= \frac{e_\gamma}{2\left(\sqrt{(x - e_\gamma}\;\sqrt{R'e_\gamma}\right)^3} + \frac{1}{\sqrt{x - e_\gamma}}\,\mathfrak{P}_1\,(x - e_\gamma)$$

$$\int \frac{x^n\,dx}{2\;Rx\,\sqrt{Rx}} = - \frac{e_\gamma^n}{\sqrt{R'^3\,e_\gamma}\sqrt{x - e_\gamma}} + \sqrt{x - e_\gamma}\;\mathfrak{P}_2\,(x - e_\gamma)$$

$$\frac{\phi(x)}{y} = \frac{\phi(e_\gamma)}{\sqrt{R'\,e_\gamma}\;\sqrt{x - e_\gamma}}\left\{1 + \mathfrak{P}_3\,(x - e_\gamma)\right\}$$

$$\int^{\prime} \frac{\phi x\, dx}{y} = \frac{\phi\, e_\gamma}{\sqrt{R' e_\gamma}} \sqrt{x - e_\gamma}\; \mathfrak{P}_1 (x - e_\gamma)$$

Quum praeterea sit $y = \sqrt{R' e_\gamma}\, \sqrt{x - e_\gamma} \left\{ 1 + \mathfrak{P}(x - e_\gamma) \right\}$

et $\psi(x\, y) = P(x) + y\, Q = y \left(\int \frac{x^n\, dx}{2 y^3} - \int \frac{\phi(x)\, dx}{y} \right)$

concluditur facile, non fieri functionem ψ infinitam, quum $x = e_\gamma$ est. Fit propter hoc functio $\psi(x\, y)$ infinita pro infinitis valoribus solum argumentorum. Omnem autem functionem hujus generis e x et y per additionem et multiplicationem componi notum est. Est igitur $P(x)$ functio integra.

Ad gradum functionis Px determinandum, quem p vocare libet, evolvamus in seriem utramque partem aequationis nostrae:

$$\frac{x^n}{2 Rx \sqrt{Rx}} - \frac{\phi(x)_n}{\sqrt{Rx}} = \frac{d}{dx} \frac{P(x)_n}{y}$$

pro magnis valoribus x. Erit:

$$\frac{1}{2} x^n \left(4 x^{2\rho+1} + g_1 x^{2\rho} + \ldots g_{2\rho+1}\right)^{-\frac{3}{2}} - \left(l_{2\rho-1} x^{2\rho-1} + l_{2\rho-2} x^{2\rho-2} + \ldots + l_0\right) \times$$

$$\times \left(4 x^{2\rho+1} + g_1 x^{2\rho} + \ldots + g_{2\rho+1}\right)^{-\frac{1}{2}} = \frac{d}{dx} \left\{ \left(p_n x^p + p_{n-1} x^{p-1} + \ldots p_0\right) \left(4x^{2\rho+1} + \ldots g_{2\rho+1}\right)^{-\frac{1}{2}} \right\}$$

vel: $$\frac{1}{2} x^{\frac{2n - 6\rho - 3}{2}} \left(4 + \frac{g_1}{x} + \ldots\right)^{-\frac{3}{2}} - x^{\frac{2\rho - 3}{2}} \left(l_{2\rho-1} + l_{2\rho-2} x^{-1} + \ldots + l_0 x^{1-2\rho}\right) \times$$

$$\times \left(4 + g_1 x^{-1} + \ldots\right)^{-\frac{1}{2}} = \frac{d}{dx}\, x^{\frac{2p - 2\rho - 1}{2}} \left(p_n + p_{n-1} x^{-1} + \ldots\right) \left(4 + g_1 x^{-1} + \ldots\right)^{-\frac{1}{2}}$$

Summae potestates variabilis x sunt resp.:

$$x^{\frac{2n - 6\rho - 3}{2}},\; x^{\frac{2\rho - 3}{2}},\; x^{\frac{2p - 2\rho - 1}{2} - 1}$$

est autem semper $2n - 6\rho - 3 < 2\rho - 3$, quum limes superior pro n sit $4\rho - 1$.

Quum summae potestates utriusque partis altera alteri aequalis esse debeat, est $\frac{2p - 2\rho - 1}{2} - 1$ necessario aequale $\frac{2\rho - 3}{2}$, vel

$$2p - 2\rho - 3 = 2\rho - 3$$

$$p = 2\rho$$

Gradus functionis $P(x)_n$ non pendet igitur a valore n.

Aequationem:

$$\frac{x^n}{2 y^3} = \frac{\phi_n(x)}{y} + \frac{2 R(x) P'_n(x) - P_n(x) R'(x)}{2 y^3}$$

sequitur haec:

$$x^n = 2(\phi_n + P'_n)R - P_n R'$$

6) vel $x^n = \overline{M}(x)_n R(x) + \overline{N}(x)_n R'(x)$

ubi $\overline{M}(x)_n = \phi(x)_n + P'(x)_n$ functio est $(2\rho - 1)^{ti}$ gradus

et $\overline{N}(x)_n = -P(x)_n$ „ „ $2\rho^{ti}$ „

n est $= O, 1 .. 4\rho - 1$; gradus functionis ad dextram est autem 4ρ. Aequatio nostra, quum sit identica, suppeditat nobis $4\rho + 1$ aequationes ad totidem incognitas quantitates determinandas, quae sunt, 2ρ coëfficientes functionis $\phi(x)_n$: $l_{n,o}, l_{n,1}, \ldots l_{n,2\rho-1}$ et $2\rho + 1$ coëfficientes functionis $P(x)_n$.

Intelligitur facile, determinantem systematis aequationum nostrarum non evanescere, quum sit illa discriminans functionis Rx, quam nullitati aequalem non esse statuimus. Problema nostrum ad has aequationes lineares solvendas revocatum est.

Multo elegantius problema functionum $\overline{M}(x)_n$ et $\overline{N}(x)_n$ destinandarum resolvitur per methodum, quam Ill. Jacobi in Diario Crelliano tomo XV. publicavit, sub titulo: De eliminatione variabilis e duabus aequationibus algebraicis. Formulas, quibus nobis opus est, breviter exponamus.

Fac, datas esse aquationes

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_o = O$$
$$\phi(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_o = O$$

Deduc de iis novas n aequationes secundum formulam:

$$m_s = (a_n x^{n-s-1} + a_{n-1} x^{n-s-2} + \ldots + a_{s+1})\phi(x) - (b_n x^{n-s-1} + b_{n-1} x^{n-s-2} + \ldots + b_{s+1}) f(x) = \alpha_{o,s} x^o + \alpha_{1,s} x^1 + \ldots + \alpha_{n-1,s} x^{n-1}$$

est autem $\alpha_{r,s} = \alpha_{s,r} = a_{s+1} b_r + a_{s+2} b_{r-1} + \ldots + a_{r+s+1} b_o$

$$- (b_{s+1} a_r + b_{s+2} a_{r-1} + \ldots + b_{r+s+1} a_o)$$

$$\left(\begin{matrix} b_\alpha \\ a_\alpha \end{matrix} \Big\{ = O \text{ si } \alpha > \text{n} \right)$$

$x^o, x^1, .. x^{n-1}$ considerentur tamquam incognitae quantitates. Per resolutionem aequationum $m_s = O, (s = O, 1, \ldots, n-1)$ reperietur:

$$Lx^r = A_{r,o} m_o + A_{r,1} m_1 + \ldots + A_{r,n-1} m_{n-1}$$

$$\text{ubi } L = \sum \pm \alpha_{oo} \alpha_{11} \ldots \alpha_{n-1,n-1}, \quad A_{r,s} = \frac{dL}{d\alpha_{r,s}}$$

Tum est $A_{s,r} = A_{r,s}$ et $A_{r,s} = A_{r',s'}$ si $r + s = r' + s'$. Propter hoc scribitur A_{r+s} loco $A_{r,s}$, est igitur:

7) $Lx^r = A_r m_o + A_{r+1} m_1 + \ldots + A_{r+n-1} m_{n-1}$ $(r = 0, 1, 2, .. n-1)$

et $L = \alpha_{r,o} A_r + \alpha_{r,1} A_{r+1} + \ldots + \alpha_{r,n-1} A_{r+n-1}$

Tum hae inter $A_1, A_2, ..$ deducuntur relationes identicae:

$$\left. \begin{matrix} a_o A_r + a_1 A_{r+1} + \ldots + a_n A_{r+n} = O \\ \text{vel etiam } b_o A_r + b_1 A_{r+1} + \ldots + b_n A_{r+n} = O \end{matrix} \right\} \quad (r = 0, 1, 2, \ldots n-2)$$

Sint $M(x)_r$, $N(x)_r$ functiones ipsius x rationales integrae $(n-1)^{ti}$ ordinis; functiones hae semper ita determinari possunt, ut sit

8) $$M(x)_r f(x)_r + N(x)_r \phi(x)_r = Lx^r \qquad \left(r \leqq 2n-1\right)$$

Si $r \leqq n-1$, substitutis in 7) expressionibus m_s ponendum esse videmus:

$$\begin{aligned} -M_r = & A_r (b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots + b_1) \\ & + A_{r+1} (b_n x^{n-2} + b_{n-1} x^{n-2} + \ldots + b_2) \\ & + \ldots . \\ & + A_{r+n-1} b_n \\ N_r = & A_r (a_n x^{n-1} + a_{n-1} x^{n-2} + \ldots + a_1) \\ & + A_{r+1} (a_n x^{n-2} + a_{n-1} x^{n-3} + \ldots + a_2) \\ & + \ldots . \\ & + A_{r+1} a_n \end{aligned}$$

Ut functiones M_r et N_r tum, quum $n < r \leqq 2n-1$ est, eruantur, ponendum est in formulis antecedentibus $\frac{1}{x}$ loco x, simulque a_{n-r}, b_{n-r} loco a_r, b_r. Quo facto

$$\begin{aligned} \text{abit} \quad & f(x)_n \text{ in } x^{-n} f(x)_n \\ \text{„} \quad & \phi(x)_n \text{ „ } x^{-n} \phi(x)_n \\ & m_r \text{ „ } -m_{n-1-r} x^{1-n} \\ & a_{r,s} \text{ „ } -a_{n-1-r,\, n-1-s} \\ & L \text{ „ } (-1)^n L \\ & A_r \text{ „ } (-1)^{n-1} A_{2n-2-r} \end{aligned}$$

et obtinetur

9) $$M_{2n-1-r} f(x) + N_{2n-1-r} \phi(x) = L x^{2n-1-r}, (r = 0, 1, 2 \ldots n-1),$$

ubi
$$\begin{aligned} M_{2n-1-r} = & A_{2n-2-r} (b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-1} x^{n-1}) \\ & + A_{2n-3-r} (b_0 + b_1 x + b_2 x^2 + \ldots + b_{n-2} x^{n-2}) x \\ & + \ldots . \\ & + A_{n-1-r} b_0 x^{n-1} \\ -N_{2n-1-r} = & A_{2n-2-r} (a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}) \\ & + A_{2n-3-r} (a_0 + a_1 x + \ldots + a_{n-2} x^{n-2}) x \\ & + \ldots . \\ & + A_{n-1-r} x^{n-1} a_0 \end{aligned}$$

Si altera functionum $\phi(x)$, $f(x)$ inferioris ordinis est, nil mutabitur, nisi quod coëfficientes potestatum inferiorum, in ea deficientium, in formulis nostris nullitati aequandae erunt.

Ponendum est nostro casu $f(x) = R(x)$ et $\phi(x) = R'(x)$. Tum erit $n = 2\rho + 1$ et

$$a_\mu = g_{2\rho+1-\mu}.$$
$$b_\nu = \overline{\nu+1}\, g_{2\rho-\nu}, \quad b_{2\rho+1} = b_n = O$$

Inveniemus

10) $$m_s = (g_0\, x^{2\rho-s} + g_1\, x^{2\rho-s-1} + \ldots + g_{2\rho-s})\, R'(x) - \left(\overline{2\rho+1} g_0\, x^{2\rho-s-1} + 2\rho g_1\, x^{2\rho-s-2} + \ldots + (s+2) g_{2\rho-s-1}\right) Rx$$
$$(s = 0, 1, 2, \ldots 2\rho)$$

vel $$m_s = \alpha_{0s} + \alpha_{1,s}\, x + \alpha_{2,s}\, x^2 + \ldots + \alpha_{2\rho,s} x^{2\rho}$$

ubi $$\alpha_{r,s} = (r+1)\, g_{2\rho-r}\, g_{2\rho-s} + r\, g_{2\rho-r+1}\, g_{2\rho-s-1} + \ldots + (r+1-n)\; g_{2\rho-r+n}\, g_{2\rho-s-n} + \ldots + g_{2\rho}\, g_{2\rho-s-r}$$
$$- \left\{ (s+2)\, g_{2\rho-r+1}\, g_{2\rho-s-1} + \ldots + (s+1-n)\; g_{2\rho-r+n}\, g_{2\rho-s-n} + \ldots + (r+s+2)\; g_{2\rho+1}\, g_{2\rho-s-r-1} \right\}$$
$$= (r+1)\, g_{2\rho-r}\, g_{2\rho-s} + \sum_{1}^{r+1}{}_n\, (r-s-2n)\, g_{2\rho-r+n}\, g_{2\rho-s-n}$$

Quum autem $\alpha_{r,s} = \alpha_{s,r}$ sit, has duas $\alpha_{r,s}$ induere potest formas:

11) $$\alpha_{r,s} = \begin{cases} = (r+1)\, g_{2\rho-r}\, g_{2\rho-s} - \sum_{1}^{r+1}{}_n\, (s-r+2n)\, g_{2\rho-r+n}\, g_{2\rho-s-n} \\ = (s+1)\, g_{2\rho-r}\, g_{2\rho-s} - \sum_{1}^{s+1}{}_n\, (r-s+2n)\, g_{2\rho-s+n}\, g_{2\rho-r-n} \end{cases} \quad \begin{cases} g_m = O \text{ si } m < O \text{ vel si } m > 2\zeta+1 \\ g_0 = 4 \end{cases}$$

quarum altera si $s \geqq r$, altera si $s \leqq r$ jam ad simplicissimam formam reducta apparet.

Ponendum nunc est $L = \sum \pm\, \alpha_{00}\, \alpha_{11} \ldots \alpha_{2\rho,2\rho}$; est autem L discriminans functionis nostrae $R(x)$. Erit igitur:

12) $$L\, x^r = A_r\, m_0 + A_{r+1}\, m_1 + \ldots + A_{r+2\rho}\, m_{2\rho} \qquad A_r = \frac{dL}{d\,\alpha_{n,r-n}}$$

et functiones A_a his satisfaciunt aequationibus:

$$O = g_{2\rho+1}\, A_s + g_{2\rho}\, A_{s+1} + g_{2\rho-1}\, A_{s+2} + \ldots + g_1\, A_{s+2\rho} + 4\, A_{s+2\rho+1}$$

vel etiam $$O = g_{2\rho}\, A_s + 2\, g_{2\rho-1}\, A_{s+1} + 3\, g_{2-2\rho} A_{s+2} + \ldots + 2\rho\, g_1\, A_{s+2\rho-1} + 4\,(2\rho+1)\, A_{s+2\rho}$$
$$(s = O, 1, \ldots 2\rho-1)$$

Omnes igitur quantitates A_a per 2ρ earum exprimuntur.

Itaque ex aequatione 6) $x^r = 2\,(\phi_r + P'_{r\,ti})\, R - P_{r\,ti}\, R' = \overline{M}_r\, R + \overline{N}_r\, R'$, ubi $\overline{M}_r$, $\overline{N}_r$ functiones quaedam integrae resp. $(2\rho-1)$ et 2ρ gradus erant, si litera L indicem omnibus coëfficientibus functionum $\overline{M}_r$ et $\overline{N}_r$ communem designamus, et

$$\overline{M_r} = \frac{M_r}{L}, \quad \overline{N_r} = \frac{N_r}{L}$$

ponimus, oritur aequatio:

13) $$L x^r = M_r R + N_r R'$$

Quae autem functiones pro M_r et N_r ponendae sint docent formulae illae e disputatione Ill. Jacobi deductae.

Si enim $r \leqq 2\rho$ est, et formulas 12) et 10)

spectaveris, reperietur:

$$\begin{aligned} - M_r = & A_r (\overline{2\rho+1} g_0 x^{2\rho-1} + 2\rho g_1 x^{2\rho-2} + \ldots + 2 g_{2\rho-1}) \\ & + A_{r+1} (\overline{2\rho+1} g_0 x^{2\rho-2} + \ldots + 3 g_{2\rho-2}) \\ & + \ldots . \\ & + A_{r+2\rho-2} (\overline{2\rho+1} g_0 x + 2\rho g_1) \\ & + A_{r+2\rho-1} \overline{2\rho+1} g_0 \end{aligned}$$

$$\begin{aligned} \text{et } N_r = & A_r (g_0 x^{2\rho} + g_1 x^{2\rho-1} + \ldots + g_{2\rho}) \\ & + A_{r+1} (g_0 x^{2\rho-1} + \ldots + g_{2\rho-1}) \\ & + \ldots . \\ & + A_{r+2\rho-1} (g_0 x + g_1) \\ & + A_{r+2\rho} g_0 \end{aligned}$$

vel etiam $- M_r = \overline{2\rho+1} g_0 . A_r x^{2\rho-1} + (\overline{2\rho+1} g_0 A_{r+1} + 2\rho g_1 A_r) x^{2\rho-2} + \ldots .$

$$+ x^{2\rho-m} (\overline{2\rho+1} g_0 A_{r+m-1} + 2\rho g_1 A_{r+m-2} + \overline{2\rho-1} g_2 A_{r+m-3}$$
$$+ \ldots + (2\rho + 2 - m) g_{m-1} A_r)$$
$$+ \ldots + (\overline{2\rho+1} g_0 A_{r+2\rho-1} + \ldots + 3 g_{2\rho-2} A_{r+1} + 2 g_{2\rho-1} A_r)$$
$$= \sum_{m=0}^{2\rho-1} \sum_{n=0}^{2\rho-1-m} (m + n + 2) g_{2\rho-1-m-n} A_{r+n} x^m$$

Simili modo erit:

$$N_r = g_0 A_r x^{2\rho} + (g_0 A_{r+1} + g_1 A_r) x^{2\rho-1} + \ldots + (g_0 A_{r+2\rho} + g_1 A_{r+2\rho-1}$$
$$+ \ldots + g_{2\rho-1} A_{r+1} + g_{2\rho} A_r)$$

$$= \sum_{m=0}^{2\rho} \sum_{n=0}^{2\rho-m} g_{2\rho-m-n} A_{r+n} x^m$$

$$\text{et } N'_r = \sum_{m=0}^{2\rho-1} \sum_{n=0}^{2\rho-m-1} (m + 1) g_{2\rho-1-m-n} A_{r+n} x^m$$

Habemus autem

$$2 (\phi_r + P'_r) = \frac{M_r}{L} \quad \text{et} \quad - P_r = \frac{N_r}{L}$$

In iis quae sequuntur functio ϕ_r tantum requiritur, quae est

$$2\,L\,\phi_r = -2\,L\,P'_r + M_r = M_r + 2\,N'_r$$

$$= -\sum_{o}^{2\rho-1}{}_m \sum_{o}^{2\rho-1-m}{}_n (m+n+2)\, g_{2\rho-1-m-n}\, A_{r+n}\, x^m$$

$$+ \sum_{o}^{2\rho-1}{}_m \sum_{o}^{2\rho-1-m}{}_n (2\,m+2)\, g_{2-1-m-n}\, A_{r+n}\, x^m$$

14) $$2\,L\,\phi_r = \sum_{o}^{2\rho-1}{}_m \sum_{o}^{2\rho-1-m}{}_n (m-n)\, g_{2\rho-1-m-n}\, A_{r+n}\, x^m$$

$(r = 0, 1, 2, \ldots 2\rho)$

Jam similem formulam pro $r > 2\rho$ explicemus.

E formulis generalibus supra constitutis facile hoc loco concluditur:

$$M_{4\rho+1-r} = A_{4\rho-r}\,(g_{2\rho} + 2\,g_{2\rho-1}\,x + 3\,g_{2\rho-2}\,x^2 + \ldots + \overline{2\rho+1}\,g_o\,x^{2\rho})$$
$$+ A_{4\rho-r-1}\,(g_{2\rho} + 2\,g_{2\rho-1} + \ldots + 2\rho\,g_1\,x^{2\rho-1})x$$
$$+ \ldots .$$
$$+ A_{2\rho-r}\,g_{2\rho}\,x^{2\rho}$$
$$= g_{2\rho}\,A_{4\rho-r} + (g_{2\rho}\,A_{4\rho-r-1} + 2\,g_{2\rho-1}\,A_{4\rho-r})x$$
$$+ \ldots + (g_{2\rho}\,A_{2\rho-r+1} + 2\,g_{2\rho-1}\,A_{2\rho-r+2} + \ldots + 2\rho\,g_o\,A_{4\rho-r})x^{2\rho-1}$$

Coëfficiens potestatis $x^{2\rho}$ semper evanescit, quod e relationibus, quae sunt inter A_α sequitur. Est enim ille:

$$\overline{2\rho+1}\,g_o\,A_{4\rho-r} + 2\rho\,g_1\,A_{4\rho-r-1} + \overline{2\rho-1}\,g_2\,A_{4\rho-r-2} + \ldots + A_{2\rho-r}\,g_{2\rho}$$

ubi $r \geqq 2$. Invenitur autem:

$$\overline{2\rho+1}\,g_o\,A_{s+2\rho} + 2\rho\,g_1\,A_{s+2\rho-1} + \ldots + 2\,g_{2\rho-1}\,A_{s+1} + g_{2\rho}\,A_s = 0$$

$(s = 0, 1, 2 \ldots 2\rho-1)$

Functio $M_{4\rho+1-r}$ etiam hoc modo scribi potest:

$$M_{4\rho+1-r} = \sum_{o}^{2\rho-1}{}_m \sum_{o}^{m}{}_n A_{4\rho-r-n}\, g_{2\rho+n-m}\,(m+1-n)x^m$$

Formula haec haudquaquam mutabitur, si summatio secundum n usque ad $n = m+1$ fit, quum coëfficiens potestatis x^{m+1} per se tollatur. Propter ea autem, quae sequuntur, haud parvi est momenti, hunc summationis limitem introducere.

Est igitur

$$M_{4\rho+1-r} = \sum_{o}^{2\rho-1} \sum_{o}^{m+1}{}_n (m+1-n)\, g_{2\rho+n-m}\, A_{4\rho-r-n}\, x^m$$

Simili modo fit

$$\begin{aligned}
-N_{4\rho+1-r} &= A_{4\rho-r}\,(g_{2\rho+1} + g_{2\rho}\,x + g_{2\rho-1}\,x^2 + \ldots + g_1\,x^{2\rho}) \\
&\quad + A_{4\rho-r-1}\,x\,(g_{2\rho+1} + g_{2\rho}\,x + \ldots + g_2\,x^{2\rho-1}) \\
&\quad + \ldots . \\
&\quad + A_{2\rho-r}\,x^{2\rho}\,g_{2\rho+1} \\
&= A_{4\rho-r}\,g_{2\rho+1} + (A_{4\rho-r}\,g_{2\rho} + A_{4\rho-r-1}\,g_{2\rho+1})\,x \\
&\quad + (A_{4\rho-r}\,g_{2\rho-1} + A_{4\rho-r-1}\,g_{2\rho} + A_{4\rho-r-2}\,g_{2\rho+1})\,x^2 \\
&\quad + \ldots + (A_{4\rho-r}\,g_1 + A_{4\rho-r-1}\,g_2 + \ldots + A_{2\rho-r}\,g_{2\rho+1})\,x^{2\rho} \\
&= \sum_{o}^{2\rho}{}_m \sum_{o}^{m}{}_n \; g_{2\rho+1-m+n}\,A_{4\rho-r-n}\,x^m
\end{aligned}$$

Tum erit:

$$-N'_{4\rho+1-r} = -\frac{d}{dx}\,N_{4\rho+1-r} = \sum_{o}^{2\rho-1}{}_m \sum_{o}^{m+1}{}_n \;(m+1)\,g_{2\rho-m+n}\,A_{4\rho-r-n}\,x^m$$

et

$$\begin{aligned}
2\,L\,\phi_{4\rho+1-r} &= M_{4\rho+1-r} + 2\,N'_{4\rho+1-r} \\
&= \sum_{o}^{2\rho-1}{}_m \sum_{o}^{m+1}{}_n \;(m+1-n)\,g_{2\rho-m+n}\,A_{4\rho-r-n}\,x^m \\
&\quad + \sum_{o}^{2\rho-1}{}_m \sum_{o}^{m+1}{}_n \;(-2m-2)\,g_{2\rho-m+n}\,A_{4\rho-r-n}\,x^m
\end{aligned}$$

15) $$2\,L\,\phi_{4\rho+1-r} = \sum_{o}^{2\rho-1}{}_m \sum_{o}^{m+1}{}_n \;-(m+n+1)\,g_{2\rho-m+n}\,A_{4\rho-r-n}\,x^m$$

$$(r = 2, 3, 4, \ldots 2\rho)$$

Profecti sumus ab aequatione 5a)

$$f_r\,(x) = \int \frac{x^r\,dx}{2\,R_x\,\sqrt{R_x}} = \int \frac{\phi_r\,(x)\,dx}{\sqrt{R_x}} + \frac{P_r\,(x) + Q_r\,y}{\sqrt{R_x}}$$

$$(r = 0, 1, \ldots 4\rho-1)$$

et reperimus, quae sit functiones ϕ_r, P_r, Q_r; restat, ut $\dfrac{\phi_r\,(x)}{\sqrt{R_x}}$ ad nostras functiones $H\,(x\,y)_\alpha$ et $H'\,(x\,y)_\alpha$ revocetur. Est autem:

$$\frac{x^{\alpha-1}}{\hbar} = H\,(x\,y)_\alpha \qquad (\alpha = 1, 2, \ldots \rho)$$

$$\frac{x^\rho}{y} = -\,H'\,(x\,y)_\rho$$

et
$$\frac{x^{\rho+a}}{y} = \sum_{\rho-a+1}^{\rho}{}_t\, g_{\rho+a+1,\,t}\; H(xy)_t + \sum_{\rho-a}^{\rho}{}_t\, g'_{\rho+a+1,\,t}\; H'(xy)_t$$
$$(a = 1, 2, \ldots, (\rho-1))$$

ergo:

I) si $r \leqq 2\rho$

$$\frac{2L\phi(x)_r}{y} = \sum_{o}^{2\rho-1}{}_m \sum_{o}^{2\rho-1-m}{}_n (m-n)\, g_{2\rho-1-m-n}\, A_{r+n}\, \frac{x^m}{y}$$

$$= \sum_{o}^{\rho-1}{}_m \sum_{o}^{2\rho-1-m}{}_n (m-n)\, g_{2\rho-1-m-n}\, A_{r+n}\, \frac{x^m}{y} + \sum_{o}^{\rho-1}{}_n (\rho-n)\, g_{\rho-1-n}\, A_{r+n}\, \frac{x^\rho}{y}$$
$$+ \sum_{1}^{\rho-1}{}_m \sum_{o}^{\rho-1-m}{}_n (\rho+m-n)\, g_{\rho-1-m-n}\, A_{r+n}\, \frac{x^{\rho+m}}{y}$$

$$= \sum_{1}^{\rho}{}_m \sum_{o}^{2\rho-m} (m-1-n)\, g_{2\rho-m-n}\, A_{r+n}\, H(xy)_m - \sum_{o}^{\rho-1}{}_n (\rho-n) g_{\rho-1-n}\, A_{r+n}\, H'(xy)_\rho$$
$$+ \sum_{1}^{\rho-1}{}_m \left[\sum_{o}^{\rho-1-m}{}_n (\rho+m-n)\, g_{\rho-1-m-n}\, A_{r+n}\right] \left\{ \sum_{\rho-m+1}^{\rho} g_{\rho+m+1,\,t}\, H(xy)_t \right.$$
$$\left. + \sum_{\rho-m}^{\rho} g'_{\rho+m+1,\,t}\, H'(xy)_t \right\}$$

Intueamur partem hanc ultimam.
Primo igitur habemus loco

16) $$\sum_{1}^{\rho-1}{}_m \left(\sum_{o}^{\rho-1-m}{}_n (\rho+m-n)\, g_{\rho-1-m-n}\, A_{r+n} \right) \sum_{\rho-m+1}^{\rho}{}_t\, g_{\rho+m+1,\,t}\, H(xy)_t$$
$$= \sum_{2}^{\rho}{}_t\, H(xy)_t \sum_{\rho-t+1}^{\rho-1}{}_m\, g_{\rho+m+1,\,t} \sum_{o}^{\rho-1-m}{}_n (\rho+m-n)\, g_{\rho-1-m-n}\, A_{r+n}$$

tum est
$$\sum_{\rho-t+1}^{\rho-1}{}_m\, g_{\rho+m+1,\,t} \sum_{o}^{\rho-1-m}{}_n (\rho+m-n)\, g_{\rho-1-m-n}\, A_{r+n}$$
$$= \sum_{o}^{t-2}{}_n\, A_{r+n} \sum_{\rho-1-n}^{\rho-t+1} (\rho+m-n)\, g_{\rho-1-m-n}\, g_{\rho+m+1,\,t}$$

habemus ergo

$$16) \quad = \sum_{2}^{\rho}{}_{t} \; H(x\,y)_t \sum_{o}^{t-2}{}_{n} A_{r+n} \sum_{\rho-1-n}^{\rho-t+1} (\rho + m - n)\, g_{\rho-1-m-n}\; g_{\rho+m+1,\,t}$$

Simili modo oritur:

$$17) \quad \sum_{1}^{\rho-1}{}_{m} \left\{ \sum_{o}^{\rho-1-m}{}_{n} (\rho + m - n)\, g_{\rho-1-m-n}\; A_{r+n} \right\} \sum_{\rho-m}^{\rho}{}_{t} \; g'_{\rho+m+1,\,t}\; H'(x\,y)_t$$

$$- \sum_{o}^{\rho-1}{}_{n} (\rho - n)\, g_{\rho-1-n}\; A_{r+n}\, H'(x\,y)_\rho$$

$$= \sum_{o}^{\rho-1}{}_{m} \sum_{o}^{\rho-1-m}{}_{n} (\rho + m - n)\, g_{\rho-1-m-n}\; A_{r+n} \sum_{\rho-m}^{\rho}{}_{t} \; g'_{\rho+m+1,\,t}\; H'(x\,y)_t \quad (g'_{\rho+1,\,\rho} = -1)$$

$$= \sum_{1}^{\rho}{}_{t} \; H'(x\,y)_t \sum_{\rho-t}^{\rho-1}{}_{m} \sum_{o}^{\rho-1-m}{}_{n} (\rho + m - n)\, g_{\rho-1-m-n}\; A_{r+n}\; g'_{\rho+m+1,\,t}$$

Habemus praeterea:

$$\sum_{\rho-t}^{\rho-1}{}_{m} \sum_{o}^{\rho-1-m}{}_{n} (\rho + m - n)\, g_{\rho-1-m-n}\; A_{r+n}\; g'_{\rho+m+1,\,t}$$

$$= \sum_{o}^{t-1}{}_{n} A_{r+n} \sum_{\rho-1-n}^{\rho-t}{}_{m} (\rho + m - n)\, g_{\rho-1-m-n}\; g'_{\rho+m+1,\,t}$$

est igitur

$$17) \quad = \sum_{1}^{\rho}{}_{t} \; H'(x\,y)_t \sum_{o}^{t-1}{}_{n} A_{r+n} \sum_{\rho-1-n}^{\rho-t}{}_{m} (\rho + m - n)\, g_{\rho-1-m-n}\; g'_{\rho+m+1,\,t}$$

et propter hoc

$$\frac{2\,L\,\phi_r}{y} = \sum_{1}^{\rho}{}_{m} \sum_{o}^{2\rho-m}{}_{n} (m - n - 1)\, g_{2\rho-m-n}\; A_{r+n}\; H(x\,y)_m$$

$$+ \sum_{2}^{\rho}{}_{t} \; H(x\,y)_t \sum_{o}^{t-2}{}_{n} A_{r+n} \sum_{\rho-1-n}^{\rho-t+1}{}_{m} (\rho + m - n)\, g_{\rho-1-m-n}\; g_{\rho+m+1,\,t}$$

$$+ \sum_{1}^{\rho}{}_{t} \; H'(x\,y)_t \sum_{o}^{t-1}{}_{n} A_{r+n} \sum_{\rho-1-n}^{\rho-t}{}_{m} (\rho + m - n)\, g_{\rho-1-m-n}\; g'_{\rho+m+1,\,t}$$

Formula haec magis etiam reduci potest. Scribatur in prima linea t pro m. Quum sit $t \leqq \rho$, est $2\rho - t \geqq \rho$ ergo $2\rho - t \geqq t$ et $2\rho - t > t - 2$. Quam ob rem summa haec tali modo scribi potest:

$$\sum_{1}^{\rho}{}_t \sum_{0}^{2\rho-t}{}_n (t-n-1)\, g_{2\rho-t-n}\, A_{r+n}\, H(xy)_t = -\sum_{0}^{2\rho-1}{}_n n\, g_{2\rho-1-n}\, A_{r+n}\, H(xy)_1$$

$$+ \sum_{2}^{\rho}{}_t \sum_{0}^{t-2}{}_n (t-n-1)\, g_{2\rho-t-n}\, A_{r+n}\, H(xy)_t$$

$$+ \sum_{2}^{\rho} \sum_{t-1}^{2\rho-t}{}_n (t-n-1)\, g_{2\rho-t-n}\, A_{r+n}\, H(xy)_t$$

Tum erit:

$$18)\quad \frac{2L\phi_r}{y} = -\sum_{0}^{2\rho-1}{}_n n\, g_{2\rho-1-n}\, A_{r+n}\, H(xy)_1$$

$$+ \sum_{2}^{\rho}{}_t \left\{ \sum_{0}^{t-2}{}_n A_{r+n} \left((t-n-1)\, g_{2\rho\;\; t-n} + \sum_{\rho-t+1}^{\rho-1\;\; n}{}_m (\rho+m-n)\, g_{\rho-1-m-n}\, \mathfrak{g}_{\rho+m+1,t} \right) + \sum_{t-1}^{2\rho-1}{}_n (t-n-1)\, g_{2\rho-t-n}\, A_{r+n} \right\} H(xy)_t$$

$$+ \sum_{1}^{\rho}{}_t H'(xy)_t \sum_{0}^{t-1}{}_n A_{r+n} \sum_{\rho-t}^{\rho-1-n}{}_m (\rho+m-n)\, g_{\rho-1-m-n}\, \mathfrak{g}'_{\rho+m+1,t}$$

II) Eadem formula pro $r' = 4\rho + 1 - r > 2\rho$ deducitur hoc modo:

Habemus (cf. 15) $\dfrac{2L\,\phi_{4\rho+1-r}}{y} = \sum_{0}^{2\rho-1}{}_m \sum_{0}^{m+1}{}_n -(m+n+1)\, g_{2\rho-m+n} A_{4\rho-r-n} \dfrac{x^m}{y}$

$$= \sum_{0}^{\rho-1}{}_m \sum_{0}^{m+1}{}_n -(m+n+1)\, g_{2\rho-m+n} A_{4\rho-r-n} \frac{x^m}{y} + \sum_{0}^{\rho+1}{}_n -(\rho+n+1)\, g_{\rho+n} A_{4\rho-r-n} \frac{x^\rho}{y}$$

$$+ \sum_{1}^{\rho-1} \sum_{0}^{\rho+m+1}{}_n -(\rho+m+n+1)\, g_{\rho-m+n}\, A_{4\rho-r-n} \frac{x^{\rho+m}}{y}$$

$$= \sum_{o}^{\rho-1}{}_m \sum_{o}^{m+1}{}_n - (m+n+1)\, g_{2\rho-m+n} A_{4\rho-r-n}\ H(xy)_{m+1}$$

$$+ \sum_{o}^{\rho+1}{}_n (\rho+n+1)\, g_{\rho+n} A_{4\rho-r-n}\ H'(x'y)_\rho$$

$$+ \sum_{1}^{\rho-1}{}_m \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n} A_{4\rho-r-n} \left\{ \sum_{\rho-m+1}^{\rho}{}_t\ \mathfrak{g}_{\rho+m+1,t}\, H(xy)_t + \sum_{\rho-m}^{\rho}{}_t\ \mathfrak{g}'_{\rho+m+1,t}\, H'(xy)_t \right\}$$

Est autem primum

$$\sum_{o}^{\rho-1}{}_m \sum_{o}^{m+1}{}_n - (m+n+1)\, g_{2\rho-m+n} A_{4\rho-r-n} H(xy)_{m+1}$$

$$= \sum_{1}^{\rho} \sum_{o}^{t}{}_n - (t+n)\, g_{2\rho+1+n-t} A_{4\rho-r-n}\ H(x\,y)_t$$

deinde:

$$19)\quad \sum_{1}^{\rho-1}{}_m \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n} A_{4\rho-r-n} \sum_{\rho-m+1}^{\rho}{}_t\ \mathfrak{g}_{\rho+m+1,t}\, H(x\,y)_t$$

$$= \sum_{2}^{\rho}{}_t\ H(xy)_t \sum_{\rho-t+1}^{\rho-1}{}_m\ \mathfrak{g}_{\rho+m+1,t} \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n} A_{4\rho-r-n}$$

quum autem sit:

$$\sum_{\rho-t+1}^{\rho-1}{}_m\ \mathfrak{g}_{\rho+m+1,t} \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n} A_{4\rho-r-n}$$

$$= \sum_{\rho-t+1}^{\rho-1}{}_m\ \mathfrak{g}_{\rho+m+1,t} \left\{ \sum_{o}^{2\rho-t+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n} A_{4\rho-r-n} + \sum_{2\rho-t+2}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n} A_{4\rho-r-n} \right\}$$

$$= \sum_{o}^{2\rho-t+1}{}_n A_{4\rho-r-n} \sum_{\rho-t+1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t}$$

$$+ \sum_{2\rho-t+2}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m-n}\, \mathfrak{g}_{\rho+m+1,t}$$

est ergo:

$$19)\quad = \sum_{2}^{\rho}{}_t\, H(xy)_t \left\{ \sum_{o}^{2\rho-t+1}{}_n A_{4\rho-r-n} \sum_{\rho-t+1}^{\rho-1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t} \right.$$

$$\left. + \sum_{2\rho-t+2}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t} \right\}$$

Tum fit:

$$20)\quad \sum_{1}^{\rho-1}{}_m \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n}\, A_{4\rho-r-n} \sum_{\rho-m}^{\rho}{}_t\, \mathfrak{g}'_{\rho+m+1,t}\, H'(xy)_t$$

$$+ \sum_{o}^{\rho+1}{}_n (\rho+n+1)\, g_{\rho+n}\, A_{4\rho-r-n}\, H'(xy)_\rho$$

$$= \sum_{o}^{\rho-1}{}_m \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n}\, A_{4\rho-r-n} \sum_{\rho-m}^{\rho}{}_t\, \mathfrak{g}'_{\rho+m+1,t}\, H'(xy)_t$$

$$(\mathfrak{g}'_{\rho+1,\rho} = -1)$$

$$= \sum_{1}^{\rho}{}_t\, H'(xy)_t \sum_{\rho-t}^{\rho-1}{}_m \mathfrak{g}'_{\rho+m+1,t} \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n}\, A_{4\rho-r-n}$$

Habemus autem

$$\sum_{\rho-t}^{\rho-1}{}_m \mathfrak{g}'_{\rho+m+1,t} \sum_{o}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n}\, A_{4\rho-r-n} =$$

$$= \sum_{\rho-t}^{\rho-1}{}_m \mathfrak{g}'_{\rho+m+1,t} \left\{ \sum_{o}^{2\rho-t}{}_n - (\rho+m+n+1)\, g_{\rho-m-n}\, A_{4\rho-r-n} \right.$$

$$\left. + \sum_{2\rho+1-t}^{\rho+m+1}{}_n - (\rho+m+n+1)\, g_{\rho-m+n}\, A_{4\rho-r-n} \right\}$$

$$= \sum_{o}^{2\rho-t}{}_n A_{4\rho-r-n} \sum_{\rho-t}^{\rho-1}{}_m - \mathfrak{g}'_{\rho+m+1,t}(\rho+m+n+1)\, g_{\rho-m+n}$$

$$+ \sum_{2\rho+1-t}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+1,t}$$

Est igitur:

$$20)\quad = \sum_{1}^{\rho}{}_t\, H'(xy)_t \left\{ \sum_{o}^{2\rho-t}{}_n A_{4\rho-r-n} \sum_{\rho-t}^{\rho-1} - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+1,t} \right.$$

$$\left. + \sum_{2\rho+1-t}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+1,t} \right\}$$

Itaque scribere possumus:

$$\frac{2\,L\,\phi_{4\rho+1-r}}{y} = \sum_{1}^{\rho}{}_t \sum_{o}^{t}{}_n - (t+n)\, g_{2\rho+1+n-t}\, A_{4\rho-r-n}\, H(xy)_t$$

$$+ \sum_{2}^{\rho}{}_t H(xy)_t \left\{ \sum_{o}^{2\rho-t+1}{}_n A_{4\rho-r-n} \sum_{\rho-t+1}^{\rho-1} - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t} \right.$$

$$\left. + \sum_{2\rho+2-t}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-t}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t} \right\}$$

$$+ \sum_{1}^{\rho}{}_t H'(xy)_t \left\{ \sum_{o}^{2\rho-t}{}_n A_{4\rho-r-n} \sum_{\rho-t}^{\rho-1} - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+1,t} \right.$$

$$\left. + \sum_{2\rho+1-t}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+1,t} \right\}$$

vel etiam:

21) $$\frac{2\,L\,\phi_{4\rho+1-r}}{y} = \sum_{o}^{1}{}_n - (n+1)\, g_{2\rho+n}\, A_{4\rho-r-n}\, H(xy)_1$$

$$+ \sum_{2}^{\rho}{}_t H(xy)_t \left\{ \sum_{o}^{t}{}_n A_{4\rho-r-n} \left| - (t+n)\, g_{2\rho+1+n-t} \right. \right.$$

$$\left. + \sum_{\rho-t+1}^{\rho-t}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t} \right|$$

$$+ \sum_{t+1}^{2\rho-t+1}{}_n A_{4\rho-r-n} \sum_{\rho-t+1}^{\rho-1} - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t}$$

$$\left. + \sum_{2\rho+2-t}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1}{}_m - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}_{\rho+m+1,t} \right\}$$

$$+ \sum_{1}^{\rho}{}_t H'(xy)_t \left\{ \sum_{o}^{2\rho-t}{}_n A_{4\rho-r-n} \sum_{\rho-t}^{\rho-1} - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+1,t} \right.$$

$$\left. + \sum_{2\rho+1-t}^{2\rho}{}_n A_{4\rho-r-n} \sum_{n-\rho-1}^{\rho-1} - (\rho+m+n+1)\, g_{\rho-m+n}\, \mathfrak{g}'_{\rho+m+n,t} \right\}$$

Ponatur brevitatis causa:

22) $$\frac{2\,L\,\phi(x)_\rho}{y} = \sum_{1}^{\rho}{}_\alpha h_{r,\alpha}\, H(xy)_\alpha + \sum_{1}^{\rho}{}_\alpha h'_{r,\alpha}\, H'(xy)_\alpha$$

erit:

22a) I) si $r \leqq 2\rho$

$$h_{r,1} = -\sum_{1}^{2\varrho-1}{}_n\, n g_{2\varrho-1-n}\, A_{r+n}$$

$$h_{r,\alpha} = \sum_{o}^{\alpha-2}{}_n\, A_{r+n} \left\{(\alpha-n-1)\, g_{2\varrho-\alpha-n} + \sum_{\varrho-\alpha+1}^{\varrho-n-1}{}_m\, (\rho+m-n)\, g_{\varrho-1-m-n}\, \mathfrak{g}_{\varrho+m+1,\alpha}\right\}$$

$$+ \sum_{\alpha-1}^{2\varrho-\alpha}{}_n\, (\alpha-n-1)\, g_{2\varrho-\alpha-n}\, A_{r+n}$$

$$(\alpha = 2,\ 3, \ldots \rho)$$

$$h'_{r,\alpha} = \sum_{o}^{\alpha-1}{}_n\, A_{r+n} \sum_{\varrho-\alpha}^{\varrho-1-n}{}_m\, (\rho+m-n))\, g_{\varrho-1-m-n}\, \mathfrak{g}'_{\varrho+m+1,\alpha}$$

$$(\alpha = 1,\ 2,\ 3, \ldots \rho)$$

II) si $\bar{r} = 4\rho + 1 - r > 2\rho \quad (r = 2,\ 3, \ldots, 2\rho)$

$$h_{4\varrho+1-r,1} = -\sum_{o}^{1}{}_n\, (n+1)\, g_{2\varrho+n}\, A_{4\varrho-r-n}$$

$$h_{4\varrho+1-r,\alpha} = -\sum_{o}^{\alpha}{}_n\, A_{4\varrho-r\ n} \Big\{(n+\alpha)\, g_{2\varrho+1+n\ \alpha}$$

$$+ \sum_{\varrho-\alpha+1}^{\varrho-1}{}_m\, (\rho+m+n+1)\, g_{\varrho-m+n}\, \mathfrak{g}_{\varrho+m+1,\alpha}\Big\}$$

$$- \sum_{\alpha+1}^{2\varrho-\alpha+1} A_{4\varrho-r-n} \sum_{\varrho-\alpha+1}^{\varrho-1}{}_m\, (\rho+m+n+1)\, \mathfrak{g}_{\varrho-m+n}\, \mathfrak{g}_{\varrho+m+1,\alpha}$$

$$- \sum_{2\varrho+2-\alpha}^{2\varrho}{}_n\, A_{4\varrho-r-n} \sum_{n-\varrho-1}^{\varrho-1}{}_m\, (\rho+m+n+1)\, \mathfrak{g}_{\varrho-m+n}\, \mathfrak{g}_{\varrho+m+1,\alpha}$$

$$(\alpha = 2,\ 3, \ldots \rho)$$

$$h'_{4\varrho+1-r,\alpha} = -\sum_{o}^{2\varrho-\alpha}{}_n\, A_{4\varrho-r-n} \sum_{\varrho-\alpha}^{\varrho-1}{}_m\, (\rho+m+n+1)\, \mathfrak{g}_{\varrho-m+n}\, \mathfrak{g}'_{\varrho+m+1,\alpha}$$

$$- \sum_{2\varrho+1-\alpha}^{2\varrho}{}_n\, A_{4\varrho-r-n} \sum_{n-\varrho-1}^{\varrho-1}{}_m\, (\rho+m+n+1)\, \mathfrak{g}_{\varrho-m+n}\, \mathfrak{g}'_{\varrho+m+1,\alpha}$$

$$(\alpha = 1, 2, 3, \ldots \rho)$$

Harum functionum ope unaquaeque functio $fr(x)$ ad integralia primae ac secundae speciei revocatur. Habemus enim (cf. 5a et 22).

$$fr(x) = \int \frac{x^r\, dx}{2Rx\sqrt{Rx}} = \int \frac{\phi r\, dx}{\sqrt{Rx}} + \frac{P(x) + yQ}{\sqrt{Rx}}$$

23) $$= \sum_1^{\varrho}{}_{\alpha} \frac{h_{r,\alpha}}{2L} \int H(xy)_\alpha\, dx + \sum_1^{\varrho}{}_{\alpha} \frac{h'_{r,\alpha}}{2L} \int H'(xy)_\alpha\, dx + \frac{P(x) + yQ}{\sqrt{Rx}}$$

Non est opus nobis formulis his, nisi forte integralia sunt „confecta“. Quum functio algebraica, quae in formulis reperitur, evanescat, oritur:

24) $$\overline{\int \frac{x^r\, dx}{2Rx\sqrt{Rx}}} = \sum_1^{\varrho}{}_{\alpha} \frac{h_{r,\alpha}}{2L} \overline{\int H(xy)_\alpha\, dx} + \sum_1^{\varrho} \frac{h'_{r,\alpha}}{2L} \overline{\int H'(xy)_\alpha\, dx}$$

$$(r = 0, 1, 2, \ldots 4\rho - 1)$$

Ex hujusmodi expressionibus variationes periodorum nostrarum constant. Formulam 4):

$$\delta \overline{\int \frac{x^a\, dx}{\sqrt{Rx}}} = - \sum_1^{2\varrho+1}{}_{\lambda} \delta g_\lambda \overline{\int \frac{x^{2\varrho+a+1-\lambda}}{2Rx} \frac{dx}{\sqrt{Rx}}}$$

$$= - \sum_o^{2\varrho}{}_{r} \delta g_{2\varrho+1-r} \overline{\int \frac{x^{r+a}}{2Rx} \frac{dx}{\sqrt{Rx}}}$$

jam statim exprimere possumus:

25) $$\delta \overline{\int \frac{x^a\, dx}{\sqrt{Rx}}} = - \sum_o^{2\varrho}{}_{r} \delta g_{2\varrho+1-r} \sum_1^{\varrho}{}_{\alpha} \left\{ \frac{h_{r+a,\alpha}}{2L} \overline{\int H(xy)_\alpha\, dx} + \frac{h'_{r+a,\alpha}}{2L} \overline{\int H'(xy)_\alpha\, dx} \right\}$$

$$= \sum_1^{\varrho}{}_{\alpha} \left\{ \overline{\int H(xy)_\alpha\, dx} \sum_o^{2\varrho}{}_{r} \frac{-h_{r+a,\alpha}\, \delta g_{2\varrho+1-r}}{2L} + \overline{\int H'(xy)_\alpha\, dx} \sum_o^{2\varrho}{}_{r} \frac{-h'_{r+a,\alpha}\, \delta g_{2\varrho+1-r}}{2L} \right\}$$

Quum sit $\int \frac{x^a\, dx}{\sqrt{Rx}} = \int H(xy)_{a+1}\, dx$, si $a = 0, 1, \ldots, (\rho - 1)$, formula nostra continet reductiones variationum omnium nostrorum integralium primae speciei. Tum ad integralia secundae speciei adeamus, iisque operam navemus.

Habemus primum

$$H'(xy)_\varrho = -\frac{x^\varrho}{y}$$

26) $$\delta \int \overline{H'(xy)_\varrho\, dx} = \sum_{o}^{2\varrho}{}_r\, \delta g_{2\varrho+1-r} \int \overline{\frac{x^{\varrho+r}}{2Rx} \frac{dx}{\sqrt{Rx}}}$$

$$= \sum_{o}^{2\varrho}{}_r\, \delta g_{2\varrho+1-r} \sum_{1}^{\varrho}{}_\alpha \left\{ \frac{h_{\varrho+r,\alpha}}{2L} \int \overline{H(xy)_\alpha\, dx} + \frac{h'_{\varrho+r,\alpha}}{2L} \int \overline{H'(xy)_\alpha\, dx} \right\}$$

$$= \sum_{1}^{\varrho}{}_\alpha \left\{ \sum_{o}^{2\varrho}{}_r \frac{h_{\varrho+r,\alpha}}{2L}\, \delta g_{2\varrho+1-r} \int \overline{H(xy)_\alpha\, dx} \right.$$

$$\left. + \sum_{o}^{2\varrho}{}_r \frac{h'_{\varrho+r,\alpha}}{2L}\, \delta g_{2\varrho+1-r} \int \overline{H'(xy)_\alpha\, dx} \right\}$$

Tum est generaliter (cf. 3a.)

$$H'(xy)_\mathfrak{a} = -\frac{1}{4} \sum_{\mathfrak{a}}^{2\varrho-\mathfrak{a}}{}_s (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{x^s}{y}$$

$$\mathfrak{a} = 1, 2, 3, \ldots (\rho-1)$$

erit

$$\delta \int \overline{H'(xy)_\mathfrak{a}\, dx} = -\frac{1}{4} \sum_{\mathfrak{a}}^{2\varrho-\mathfrak{a}}{}_s \left\{ (s-\mathfrak{a}+1)\, \delta g_{2\varrho-\mathfrak{a}-s} \int \overline{\frac{x^s\, dx}{y}} \right.$$

$$\left. + (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s}\, \delta \int \overline{\frac{x^s\, dx}{y}} \right\}$$

$$= -\frac{1}{4} \sum_{\mathfrak{a}}^{\varrho-1}{}_s (s-\mathfrak{a}+1)\, \delta g_{2\varrho-\mathfrak{a}-s} \int \overline{\frac{x^s\, dx}{y}} - \frac{1}{4} (\rho-\mathfrak{a}+1)\, \delta g_{\varrho-\mathfrak{a}} \int \overline{\frac{x^\varrho\, dx}{y}}$$

$$- \frac{1}{4} \sum_{1}^{\varrho-\mathfrak{a}}{}_s (s-\mathfrak{a}+1)\, \delta g_{\varrho-\mathfrak{a}-s} \int \overline{\frac{x^{\varrho+s}\, dx}{y}}$$

$$- \frac{1}{4} \sum_{\mathfrak{a}}^{2\varrho-\mathfrak{a}}{}_s (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \sum_{o}^{2\varrho}{}_r - \delta g_{2\varrho+1-r} \int \overline{\frac{x^{s+r}}{2Rx} \frac{dx}{\sqrt{Rx}}}$$

$$= -\frac{1}{4} \sum_{\mathfrak{a}}^{\varrho-1}{}_s (s-\mathfrak{a}+1)\, \delta g_{2\varrho-\mathfrak{a}-s} \int \overline{H(xy)_{s+1}\, dx}$$

$$+ \frac{1}{4} (\rho-\mathfrak{a}+1)\, \delta g_{\varrho-\mathfrak{a}} \int \overline{H'(xy)_\varrho\, dx} -$$

$$- \frac{1}{4} \sum_{1}^{\varrho - \alpha}{}_{s} (\rho + s - \alpha + 1)\, \delta g_{\varrho - \alpha - s} \left\{ \sum_{\varrho - s + 1}^{\varrho}{}_{t}\, \mathfrak{g}_{\varrho + s + 1, t} \int \overline{H(xy)_t}\, dx \right.$$

$$\left. + \sum_{\varrho - s}^{\varrho}{}_{t}\, \mathfrak{g}'_{\varrho + s + 1, t} \int \overline{H'(xy)_t}\, dx \right\}$$

$$+ \frac{1}{4} \sum_{\alpha}^{2\varrho - \alpha}{}_{s} (s - \alpha + 1)\, g_{2\varrho - \alpha - s} \sum_{o}^{2\varrho}{}_{r}\, \delta g_{2\varrho + 1 - r} \sum_{1}^{\varrho}{}_{\alpha} \left\{ \frac{h_{r + s, \alpha}}{2L} \int \overline{H(xy)_\alpha}\, dx \right.$$

$$\left. + \frac{h'_{r + s, \alpha}}{2L} \int \overline{H'(xy)_\alpha}\, dx \right\}$$

Jam primum est

$$\sum_{\alpha}^{\varrho - 1}{}_{s} (s - \alpha + 1)\, \delta g_{2\varrho - \alpha - s} \int \overline{H(xy)_{s+1}}\, dx = \sum_{\alpha + 1}^{\varrho}{}_{\alpha} (a - \alpha)\, \delta g_{2\varrho + 1 - \alpha\ \alpha} \int \overline{H(xy)_\alpha}\, dx \}$$

deinde

$$\sum_{1}^{\varrho - \alpha}{}_{s} (\rho + s - \alpha + 1)\, \delta g_{\varrho - \alpha - s} \sum_{\varrho - s + 1}^{\varrho}{}_{t}\, \mathfrak{g}_{\varrho + s + 1, t} \int \overline{H(xy)_t}\, dx$$

$$= \sum_{\alpha + 1}^{\varrho}{}_{\alpha} \int \overline{H(xy)_\alpha}\, dx \sum_{\varrho - \alpha + 1}^{\varrho - \alpha}{}_{s} (\rho + s - \alpha + 1)\, \delta g_{\varrho - \alpha - s}\, \mathfrak{g}_{\varrho + s + 1, \alpha}$$

Tum

$$\sum_{\alpha}^{2\varrho - \alpha}{}_{s} (s - \alpha + 1)\, g_{2\varrho - \alpha - s} \sum_{o}^{2\varrho}{}_{r}\, \delta g_{2\varrho + 1 - r} \sum_{1}^{\varrho}{}_{\alpha} \frac{h_{r + s, \alpha}}{2L} \int \overline{H(xy)_\alpha}\, dx$$

$$= \sum_{1}^{\varrho}{}_{\alpha} \int \overline{H(xy)_\alpha}\, dx \sum_{o}^{2\varrho}{}_{r}\, \delta g_{2\varrho + 1 - r} \sum_{\alpha}^{2\varrho - \alpha}{}_{s} (s - \alpha + 1)\, g_{2\varrho - \alpha - s} \frac{h_{r + s, \alpha}}{2L}$$

Tum simili modo:

$$\sum_{1}^{\varrho - \alpha}{}_{s} (\rho + s - \alpha + 1)\, \delta g_{\varrho - \alpha - s} \sum_{\varrho - s}^{\varrho}{}_{\alpha}\, \mathfrak{g}'_{\varrho + s + 1, \alpha} \int \overline{H'(xy)_\alpha}\, dx$$

$$- (\rho - \alpha + 1)\, \delta g_{\varrho - \alpha} \int \overline{H'(xy)_\varrho}\, dx$$

$$= \sum_{o}^{\varrho - \alpha}{}_{s} (\rho + s - \alpha + 1)\, \delta g_{\varrho - \alpha - s} \sum_{\varrho - s}^{\varrho}{}_{\alpha}\, \mathfrak{g}'_{\varrho + s + 1, \alpha} \int \overline{H'(xy)_\alpha}\, dx \qquad (\mathfrak{g}'_{\varrho + 1, \varrho} = -1)$$

$$= \sum_{1}^{\varrho}{}_{\alpha} \int \overline{H'(xy)_\alpha}\, dx \sum_{\varrho - \alpha}^{\varrho - \alpha}{}_{s} (\rho + s - \alpha + 1)\, \delta g_{\varrho - \alpha - s}\, \mathfrak{g}'_{\varrho + s + 1, \alpha}$$

Denique

$$\sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{\alpha\,1}^{\varrho} \left\{ \frac{h'_{r+s,\alpha}}{2L} \int \overline{H'(x\,y)_\alpha\, dx} \right\}$$

$$= \sum_{\alpha\,1}^{\varrho} \int \overline{H'(x\,y)_\alpha} \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{h'_{r+s,\alpha}}{2L}.$$

Substitutis hisce formis jam haec oritur formula:

$$\delta \int \overline{H'(xy)_\mathfrak{a}\, dx} = -\frac{1}{4} \sum_{\alpha\,\mathfrak{a}+1}^{\varrho} (\alpha - \mathfrak{a})\, \delta g_{2\varrho+1-\mathfrak{a}-\alpha} \int \overline{H(xy)_\alpha\, dx}$$

$$-\frac{1}{4} \sum_{\alpha\,\mathfrak{a}+1}^{\varrho} \int \overline{H(x\,y)_\alpha\, dx} \sum_{s\,\varrho-\alpha+1}^{\varrho-\mathfrak{a}} (\rho+s-\mathfrak{a}+1)\, \delta g_{\varrho-\mathfrak{a}-s}\, \mathfrak{g}_{\varrho+s+1,\alpha}$$

$$+\frac{1}{4} \sum_{\alpha\,1}^{\varrho} \int \overline{H(xy)_\alpha\, dx} \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{h_{r+s,\alpha}}{2L}$$

$$-\frac{1}{4} \sum_{\alpha\,\mathfrak{a}}^{\varrho} \int \overline{H'(x\,y)_\alpha\, dx} \sum_{s\,\varrho-\alpha}^{\varrho-\mathfrak{a}} (\rho+s-\mathfrak{a}+1)\, \delta g_{\varrho-\mathfrak{a}-s}\, \mathfrak{g}'_{\varrho+s+1,\alpha}$$

$$+\frac{1}{4} \sum_{\alpha\,1}^{\varrho} \int \overline{H'(xy)_\alpha\, dx} \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{h'_{r+s,\alpha}}{2L}$$

vel

27) $$\delta \int \overline{H'(xy)_\mathfrak{a}\, dx}$$

$$= \frac{1}{4} \sum_{\varrho\,1}^{\mathfrak{a}} \int \overline{H(xy)_\alpha\, dx} \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{h_{r+s,\alpha}}{2L}$$

$$-\frac{1}{4} \sum_{\alpha\,\mathfrak{a}+1}^{\varrho} \int \overline{H(x\,y)_\alpha\, dx} \left\{ \begin{array}{l} (\alpha-\mathfrak{a})\, \delta g_{2\varrho+1-\mathfrak{a}-\alpha} \\ \quad + \sum_{s\,\varrho-\alpha+1}^{\varrho-\mathfrak{a}} (\rho+s-\mathfrak{a}+1)\, \delta g_{\varrho-\mathfrak{a}-s}\, \mathfrak{g}_{\varrho+s+1,\alpha} \\ \quad - \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{h_{r+s,\alpha}}{2L} \end{array} \right\}$$

$$+\frac{1}{4} \sum_{\alpha\,1}^{\mathfrak{a}-1} \int \overline{H'(x\,y)_\alpha\, dx} \sum_{r\,o}^{2\varrho} \delta g_{2\varrho+1-r} \sum_{s\,\mathfrak{a}}^{2\varrho-\mathfrak{a}} (s-\mathfrak{a}+1)\, g_{2\varrho-\mathfrak{a}-s} \frac{h'_{r+s,\alpha}}{2L} -$$

$$-\frac{1}{4}\sum_{\alpha}^{\rho}\int H'(xy)_\alpha\, dx \left\{\sum_{s\;\rho-\alpha}^{\rho-\alpha}(\rho+s-\alpha+1)\,\delta g_{\rho-\alpha-s}\, g'_{\rho+s+1,\alpha} - \sum_{r\;\alpha}^{2\rho}\delta g_{2\rho+1-r}\sum_{s\;\alpha}^{2\rho-\alpha}(s-\alpha+1)\,g_{2\rho-\alpha-s}\frac{h'_{r+s,\alpha}}{2L}\right\}$$

Introductis signis

$$J_\nu = \int H(xy)_\nu\, dx,\quad J'_\nu = \int H'(xy)_\nu\, dx$$

jam aequationes, quas quaesivimus, in forma finali exhibere possumus. Sunt emin hae:

$$\delta J_\beta = \sum_{1}^{\rho}{}_{\alpha}\left\{\sum_{0}^{2\rho}{}_{r}\frac{-h_{r+\beta-1,\alpha}}{2L}\,\delta g_{2\rho+1-r}\right\}J_\alpha + \sum_{1}^{\rho}{}_{\alpha}\left\{\sum_{0}^{2\rho}{}_{r}\frac{-h'_{r+\beta-1,\alpha}}{2L}\,\delta g_{2\rho+1-r}\right\}J'_\alpha$$

$$(\beta=1, 2, \ldots \rho)$$

$$\delta J'_\rho = \sum_{1}^{\rho}{}_{\alpha}\left\{\sum_{0}^{2\rho}{}_{r}\frac{h_{r+\rho,\alpha}}{2L}\,\delta g_{2\rho+1-r}\right\}J_\alpha + \sum_{1}^{\rho}\left\{\sum_{0}^{2\rho}\frac{h'_{\rho+r,\alpha}}{2L}\,\delta g_{2\rho+1-r}\right\}J'_\alpha$$

28)
$$\delta J_\beta = \frac{1}{4}\sum_{1}^{\beta}{}_{\alpha}\left\{\sum_{0}^{2\rho}\delta g_{2\rho+1-r}\sum_{s\;\beta}^{2\rho-\beta}(s-\beta+1)\,g_{2\rho-\beta-s}\frac{h_{r+s,\alpha}}{2L}\right\}J_\alpha$$

$$+\frac{1}{4}\sum_{\beta+1}^{\rho}{}_{\alpha}\left\{(\beta-\alpha)\,\delta g_{2\rho+1-\alpha-\beta} - \sum_{s\;\rho-\alpha+1}^{\rho-\beta}(\rho+1+s-\beta)\,\delta g_{\rho-\beta-s}\, g_{\rho+s+1,\alpha} + \sum_{0}^{2\rho}{}_{r}\,\delta g_{2\rho+1-r}\sum_{s\;\beta}^{2\rho-\beta}(s-\beta+1)\,g_{2\rho-\beta-s}\frac{h_{r+s,\alpha}}{2L}\right\}J_\alpha$$

$$+\frac{1}{4}\sum_{1}^{\beta-1}{}_{\alpha}\left\{\sum_{0}^{2\rho}{}_{r}\,\delta g_{2\rho+1-r}\sum_{s\;\beta}^{2\rho-\beta}(s-\beta+1)\,g_{2\rho-\beta-s}\frac{h'_{r+s,\alpha}}{2L}\right\}J'_\alpha$$

$$+\frac{1}{4}\sum_{\beta}^{\rho}{}_{\alpha}\left\{-\sum_{s\;\rho-\alpha}^{\rho-\beta}(\rho+1+s-\beta)\,\delta g_{\rho-\beta-s}\, g'_{\rho+s+1,\alpha} + \sum_{0}^{2\rho}{}_{r}\,\delta g_{2\rho+1-r}\sum_{s\;\beta}^{2\rho-\beta}(s-\beta+1)\,g_{2\rho-\beta-s}\frac{h'_{r+s,\alpha}}{2L}\right\}J'_\alpha$$

(Summa $\sum\limits_{1}^{\beta-1}$ ommittenda est, si $\beta=1$)

Valores quantitatum $h_{r,\alpha}$ scripti sunt pag. 23.

Ex formulis his facile pro omni ρ speciales formulae derivantur. Quod pro $\rho=2$ et $\rho=3$ nunc facere in animo est.

$\rho=2.$

$$\delta J_\beta = \sum_1^2{}_\alpha \left\{ \sum_o^4{}_r \frac{-h_{r+\beta-1,\alpha}}{2L} \delta g_{5-r} \right\} J_\alpha + \sum_1^2{}_\alpha \left\{ \sum_o^4{}_r \frac{-h'_{r+\beta-1,\alpha}}{2L} \delta g_{5-r} \right\} J'_\alpha$$

$(\beta=1,2)$

$$\delta J'_2 = \sum_1^2{}_\alpha \left\{ \sum_o^4{}_r \frac{h_{2+r,\alpha}}{2L} \delta g_{5-r} \right\} J_\alpha + \sum_1^2 \left\{ \sum_o^4{}_r \frac{h'_{2+r,\alpha}}{2L} \delta g_{5-r} \right\} J'_\alpha$$

$$\delta J'_1 = \frac{1}{4} \left\{ \sum_o^4{}_r \delta g_{5-r} \sum_1^3{}_s s g_{3-s} \frac{h_{r+s,1}}{2L} \right\} J_1$$

$$- \frac{1}{4} \left\{ \delta g_2 - \sum_o^4{}_r \delta g_{5-r} \sum_1^3{}_s s g_{3-s} \frac{h_{r+s,2}}{2L} \right\} J_2$$

$$- \frac{1}{4} \left\{ -2\,\delta g_1 - \sum_o^4{}_r \delta g_{5-r} \sum_1^3{}_s s g_{3-s} \frac{h'_{r+s,2}}{2L} \right\} J'_2$$

$$+ \frac{1}{4} \left\{ \sum_o^4{}_r \delta g_{5-r} \sum_1^3{}_s s g_{3-s} \frac{h'_{r+s,1}}{2L} \right\} J'_1$$

Habemus autem:

$$y^2 = 4x^5 + g_1 x^4 + g_2 x^3 + g_3 x^2 + g_4 x + g_5 \qquad \text{(Cf. pag. 7.)}$$

$$H(xy)_1 = \frac{1}{y} \qquad H'(xy)_1 = -\frac{1}{4}\,\frac{12x^3 + 2g_1 x^2 + g_2 x}{y}$$

$$H(xy)_2 = \frac{x}{y} \qquad H'(xy)_2 = -\frac{x^2}{y}$$

et reciproce:

$$\frac{x^2}{y} = -H'(xy)_2 = \mathfrak{g}'_{3,2}\, H'(xy)_2$$

$$\frac{x^3}{y} = -\tfrac{1}{3} H'(xy)_1 - \tfrac{1}{6} g_1 \frac{x^2}{y} - \tfrac{1}{12} g_2 \frac{x}{y}$$

$$= -\tfrac{1}{3} H'(xy)_1 + \tfrac{1}{6} g_1 H'(xy)_2 - \tfrac{1}{12} g_2 H(xy)_2$$

$$= \mathfrak{g}'_{4,1} H'(xy)_1 + \mathfrak{g}'_{4,2} H'(xy)_2 + \mathfrak{g}_{4,2} H(xy)_2$$

Est igitur

$$\mathfrak{g}'_{4,1} = -\tfrac{1}{3} \qquad \mathfrak{g}'_{3,2} = -1, \qquad \mathfrak{g}_{4,2} = \tfrac{-1}{12} g_2$$
$$\mathfrak{g}'_{4,2} = \tfrac{1}{6} g_1$$

tum fit (cf. 22a)

I) $r \leqq 4$

$$h_{r,1} = -\sum_{1}^{3} n g_{3-n} A_{r+n} = -g_2 A_{r+1} - 2 g_1 A_{r+2} - 12 A_{r+3}$$

$$h_{r,2} = A_r (g_2 + 12\, \mathfrak{g}_{4,2}) + \sum_{1}^{2} (1-n) g_{2-n} A_{r+n} = -4 A_{r+2}$$

$$h'_{r,2} = A_r (2 g'_1 \mathfrak{g}_{3,2} + 3.4\, \mathfrak{g}'_{4,2}) + A_{r+1}\, 4\, \mathfrak{g}'_{3,2} = -4 A_{r+1}$$

$$h'_{r,1} = A_r (3.\, 4\, \mathfrak{g}'_{4,1}) = -\tfrac{1}{3}\, 3.\, 4\, A_r = -4 A_r$$

II) $$h_{9-r,1} = -\sum_{0}^{1} (n+1) g_{4+n} A_{8-r-n} = -g_4 A_{8-r} - 2 g_5 A_{7-r}$$

$$h_{9-r,2} = -\sum_{0}^{2} A_{8-r-n} \left\{ (2+n) g_{3+n} + (4+n) g_{1+n} \mathfrak{g}_{4,2} \right\}$$

$$- A_{5-r} (7 g_4 \mathfrak{g}_{4,2}) - A_{4-r} (8 g_5 \mathfrak{g}_{4,2})$$

$$= \frac{1}{12} \left\{ (4 g_1 g_2 - 24 g_3) A_{8-r} + (5 g_2^2 - 36 g_4) A_{7-r} \right.$$

$$\left. + (6 g_2 g_3 - 48 g_5) A_{6-r} + 7 g_2 g_4 A_{5-r} + 8 g_2 g_5 A_{4-r} \right\}$$

$$h'_{9-r,2} = -\sum_{0}^{2} A_{8-n-n} \sum_{0}^{1} (3+m+n)\, g_{\substack{2\\-m+n}}\, \mathfrak{g}'_{3+m,2}$$

$$-\sum_{3}^{4} A_{8-r-n} \sum_{n-3}^{1} (3+m+n) g_{2-m+n} \mathfrak{g}'_{3+m,2}$$

$$= A_{8-r} (3 g_2 - \tfrac{4}{6} g_1^2) + A_{7-r} (4 g_3 - \tfrac{5}{6} g_1 g_2) + A_{6-r} (5 g_4 - \tfrac{6}{6} g_1 g_3)$$

$$+ A_{5-r} (6 g_5 - \tfrac{7}{6} g_1 g_4) - A_{4-r} \tfrac{8}{6} g_1 g_5$$

$$h'_{9-r,1} = -\sum_{0}^{3} A_{8-r-n} (4+n) g_{1+n} \mathfrak{g}'_{4,1} - A_{4-r} (8 g_5 \mathfrak{g}'_{4,1})$$

$$= \tfrac{1}{3} (4 g_1 A_{8-r} + 5 g_2 A_{7-r} + 6 g_3 A_{6-r} + 7 g_4 A_{5-r} + 8 g_5 A_{4-r})$$

Quantitates A_α et L constant ex $a_{r,s}$, quod est:

$$a_{r,s} = \begin{cases} = (r+1) g_{4-r} g_{4-s} - (s-r+2) g_{5-r} g_{3-s} - (s-r+4) g_{6-r} g_{2-s} \\ \quad - (s-r-6) g_{7-r} g_{1-s} - (s-r+8) g_{8-r} g_{0-s} & (s \geqq r) \\ = (s+1) g_{4-r} g_{4-s} - (r-s+2) g_{5-s} g_{3-r} - (4+r-s) g_{6-s} g_{2-r} \\ \quad - (6+r-s) g_{7-s} g_{1-r} - (8+r-s) g_{8-s} g_{0-r} & (s < r) \end{cases}$$

Tum est $L = \sum \pm a_{00}\, a_{11} \cdots a_{44}$; $\qquad A_{r+s} = \dfrac{dL}{\delta a_{r,s}}$

Inter functiones A_α has habemus relationes:

$$O = g_5 A_r + g_4 A_{r+1} + g_3 A_{r+2} + g_2 A_{r+3} + g_1 A_{r+4} + 4 A_{r+5}$$

vel etiam

$$O = g_4 A_r + 2 g_3 A_{r+1} + 3 g_2 A_{r+2} + 4 g_1 A_{r+3} + 20 A_{r+4}$$
$$r = 1, 1, 2, 3$$

$$L = \begin{vmatrix} g_4^2 - 2 g_5 g_3, & g_4 g_3 - 3 g_5 g_2 & g_4 g_2 - 4 g_5 g_1 & g_4 g_1 - 20 g_5, & 4 g_4 \\ g_4 g_3 - 3 g_5 g_2, & 2 g_3^2 - 2 g_4 g_2 - 4 g_5 g_1, & 2 g_3 g_2 - 3 g_4 g_1 - 20 g_5, & 2 g_3 g_1 - 16 g_4, & 8 g_3 \\ g_4 g_2 - 4 g_5 g_1, & 2 g_3 g_2 - 3 g_4 g_1 - 20 g_5, & 3 g_2^2 - 2 g_3 g_1 - 16 g_4 & 3 g_2 g_1 - 12 g_3, & 12 g_2 \\ g_4 g_1 - 20 g_5, & 2 g_3 g_1 - 16_4 g, & 3 g_2 g_1 - 12 g_3 & 4 g_1^2 - 8 g_2 & 16 g_1 \\ 4 g_4, & 8 g_3 & 12 g_2 & 16 g_1 & 80 \end{vmatrix}$$

Quum $g_1 = O$ ponitur, calculis peractis hi reperientur valores:

$$\begin{aligned} L = 256 \Big\{ & 27 g_2^5 g_5^2 - 18 g_2^4 g_3 g_4 g_5 + 4 g_2^4 g_4^3 + 4 g_2^3 g_3^3 g_5 - g_2^3 g_3^2 g_4^2 - 900_2^3 g_4 g_5^2 \\ & + 825 g_2^2 g_3^2 g_5^2 + 560 g_2^2 g_3 g_4^2 g_5 - 128 g_2^2 g_4^4 \\ & - 438 g_2 g_3^3 g_4 g_5 + 144 g_2 g_3^2 g_4^3 - 15000 g_2 g_3 g_5^3 + 8000 g_2 g_4^2 g_5^2 \\ & + 108 g_3^5 g_5 - 27 g_3^4 g_4^2 + 9000 g_3^2 g_4 g_5^2 \\ & - 6400 g_3 g_4^3 g_5 + 1024 g_4^5 + 60000 g_5^4 \Big\} \end{aligned}$$

$$\begin{aligned} A_0 = -256 \Big\{ & -6 g_2^4 g_4 + 2 g_2^3 g_3^2 + 176 g_2^2 g_4^2 + 80 g_2^2 g_3 g_5 - 234 g_2 g_3^2 g_4 - 1000 g_2 g_5^2 \\ & + 2400 g_3 g_4 g_5 + 54 g_3^4 + 1280 g_4^3 \Big\} \end{aligned}$$

$$\begin{aligned} A_1 = -256 \Big\{ & 9 g_2^4 g_5 - g_2^3 g_3 g_4 - 260 g_2^2 g_4 g_5 + 195 g_2 g_3^2 g_5 + 48 g_2 g_3 g_4^2 \\ & - 27 g_3^3 g_4 - 1500 g_3 g_5^2 + 1600 g_4^2 g_5 \Big\} \end{aligned}$$

$$\begin{aligned} A_2 = -256 \Big\{ & -6 g_2^3 g_3 g_5 + 2 g_2^3 g_4^2 + 150 g_2^2 g_5^2 + 20 g_2 g_3 g_4 g_5 - 32 g_2 g_4^3 \\ & - 90 g_3^3 g_5 - 2000 g_4 g_5^2 + 36 g_3^2 g_4^2 \Big\} \end{aligned}$$

$$\begin{aligned} A_3 = -256 \Big\{ & -3 g_2^3 g_4 g_5 + 4 g_2^2 g_3^2 g_5 - g_2^2 g_3 g_4^2 - 425 g_2 g_3 g_5^2 + 80 g_2 g_4^2 g_5 \\ & + 165 g_3^2 g_4 g_5 - 48 g_3 g_4^3 + 2500 g_5^3 \Big\} \end{aligned}$$

$$\begin{aligned} A_4 = -128 \Big\{ & -45 g_2^3 g_5^2 + 38 g_2^2 g_3 g_4 g_5 - 8 g_2^2 g_4^3 - 12 g_2 g_3^3 g_5 + 700 g_2 g_4 g_5^2 \\ & + 3 g_2 g_3^2 g_4^2 + 300 g_3^2 g_5^2 - 560 g_3 g_4^2 g_5 + 128 g_4^4 \Big\} \end{aligned}$$

$$\begin{aligned} A_5 = -64 \Big\{ & 195 g_2^2 g_3 g_5^2 + 4 g_2^2 g_4^2 g_5 - 146 g_2 g_3^2 g_4 g_5 - 1500 g_2 g_5^3 + 32 g_2 g_3 g_4^3 \\ & + 36 g_3^4 g_5 - 9 g_3^3 g_4^2 - 1100 g_3 g_4 g_5^2 - 320 g_4^3 g_5 \Big\} \end{aligned}$$

Tum erit autem:

$$h_{r,1} = - g_2 A_{r+1} - 12 A_{r+3}, \qquad h'_{r,1} = -4 A_r$$
$$h_{r,2} = -4 A_{r+2} \qquad\qquad h'_{r,2} = -4 A_{r+1} \qquad r = 0\ 1, 2, 3, 4$$

$$h_{9-r,1} = - g_4 A_{8-r} - 2 g_5 A_{7-r}$$

$$h_{9-r,2} = \frac{1}{12} \Big\{ -24 g_3 A_{8-r} + (5 g_2^2 - 36 g_4) A_{7-r} + (6 g_2 g_3 - 48 g_5) A_{6-r} + 7 g_2 g_4 A_{5-r} + 8 g_2 g_5 A_{4-r} \Big\}$$

$$h'_{9-r,2} = 3 g_2 A_{8-r} + 4 g_3 A_{7-r} + 5 g_4 A_{6-r} + 6 g_5 A_{4-r}$$

$$h'_{9-r,1} = \frac{1}{3} \Big\{ 5 g_2 A_{7-r} + 6 g_3 A_{6-r} + 7 g_4 A_{5-r} + 8 g_5 A_{4-r} \Big\}$$

$$(r = 3, 4)$$

et etiam:

$$\delta J_\beta = \sum_1^2{}_a \left\{ \sum_0^3{}_r \frac{-h_{r+\beta-1,a}}{2L} \delta g_{5-r} J_a + \sum_0^3{}_r \frac{-h'_{r+\beta-1,a}}{2L} \delta g_{5-r} J'_a \right\}$$

$\beta = 1, 2$

$$\delta J'_2 = \sum_1^2{}_a \left\{ \sum_0^3{}_r \frac{h_{2+r,a}}{2L} \delta g_{5-r} J_a + \sum_0^3{}_r \frac{h'_{2+r,a}}{2L} \delta g_{5-r} J'_a \right\}$$

$$\delta J'_1 = \frac{1}{4} \sum_0^3{}_r \delta g_{5-r} \frac{g_2 h_{r+1,1} + 12 h_{r+3,1}}{2L} J_1$$

$$- \frac{1}{4} \left\{ \delta g_2 - \sum_0^3{}_r \delta g_{5-r} \frac{12 h_{r+3,2} + g_2 h_{r+1,2}}{2L} \right\} J_2$$

$$+ \frac{1}{4} \sum_0^3{}_r \delta g_{5-r} \frac{12 h'_{r+3,2} + g_2 h'_{r+1,2}}{2L} J'_2 + \frac{1}{4} \sum_0^4{}_r \delta g_{5-r} \frac{g_2 h'_{r+1,1} + 12 h'_{r+3,1}}{2L} J'_1$$

ubi $J_1 = \int \frac{dx}{y}$, $J_2 = \int \frac{x\,dx}{y}$, $J'_2 = \int \frac{-x^2\,dx}{y}$, $J'_1 = \int -\frac{1}{4} \frac{4x^3 + g_2 x}{y} dx$

$\rho = 3.$

Supponitur

$$y^2 = R(x) = 4x^7 + g_1 x^6 + g_2 x^5 + g_3 x^4 + g_4 x^3 + g_5 x^2 + g_6 x + g_7$$

et (Cf. pag. 7)

$$H(xy)_1 = \frac{1}{y} \qquad H'(xy)_1 = -\frac{1}{4} \frac{20x^5 + 4g_1 x^4 + 3g_2 x^3 + 2g_3 x^2 + g_4 x}{y}$$

$$H(xy)_2 = \frac{x}{y} \qquad H'(xy)_2 = -\frac{1}{4} \frac{12x^4 + 2g_1 x^3 + g_2 x^2}{y}$$

$$H(xy)_3 = \frac{x^2}{y} \qquad H'(xy)_3 = -\frac{x^3}{y}$$

Reciproce erit:

$$\frac{1}{y} = H(xy)_1, \quad \frac{x}{y} = H(xy)_2, \quad \frac{x^2}{y} = H(xy)_3, \quad \frac{x^3}{y} = -H'(xy)_3 = \mathfrak{g}'_{4,3} H'(xy)_3$$

$$\frac{x^4}{y} = -\frac{1}{3} H'(xy)_2 + \frac{g_1}{6} H'(xy)_3 - \frac{1}{12} g_2 H(xy)_3 = \mathfrak{g}'_{5,2} H(xy)_2 + \mathfrak{g}'_{5,2} H(xy)_3 + \mathfrak{g}_{5,3} H(xy)_3$$

$$\frac{x^5}{y} = -\frac{1}{5} H'(xy)_1 - \frac{g_1}{5} \left(-\frac{1}{3} H'(xy)_2 + \frac{g_1}{6} H'(xy)_3 - \frac{1}{12} g_2 H(xy)_3 \right) + \frac{3}{20} g_2 H'(xy)_3 - \frac{1}{10} g_3 H(xy)_3 - \frac{g_4}{20} H(xy)_2$$

$$= \mathfrak{g}'_{6,3}\, H'(xy)_3 + \mathfrak{g}'_{6,2}\, H'(xy)_2 + \mathfrak{g}'_{6,1}\, H'(xy)_1 + \mathfrak{g}_{6,3}\, H(xy)_3 + \mathfrak{g}_{6,2}\, H(xy)_2$$

est igitur

$$\mathfrak{g}'_{4,3} = -1,\ \mathfrak{g}'_{5,2} = -\frac{1}{3},\ \mathfrak{g}'_{5,3} = \frac{g_1}{6},\ \mathfrak{g}_{5,3} = -\frac{1}{12} g_2$$

$$\mathfrak{g}'_{6,1} = -\frac{1}{5},\ \mathfrak{g}'_{6,2} = \frac{1}{15} g_1,\ \mathfrak{g}'_{6,3} = \frac{9g_2 - 2g_1^2}{60},\ \mathfrak{g}_{6,3} = \frac{g_1 g_2 - 6 g_3}{60},\ \mathfrak{g}_{6,2} = -\frac{g_4}{20}$$

Tum erit

$$\delta J_\beta = \sum_1^3{}_\alpha \left\{ \sum_o^6{}_r \frac{-h_{r+\beta-1,\alpha}}{2L}\, \delta g_{7-r} J_\alpha + \sum_o^6{}_r \frac{-h'_{r+\beta-1,\alpha}}{2L}\, \delta g_{7-r}\, J'_\alpha \right\}$$

$$(\beta = 1, 2, 3)$$

$$\delta J'_3 = \sum_1^3{}_\alpha \left\{ \sum_o^6{}_r \frac{h_{r+3,\alpha}}{2L}\, \delta g_{7-r} J_\alpha + \sum_o^6{}_r \frac{h'_{r+3,\alpha}}{2L}\, \delta g_{7-r} J'_\alpha \right\}$$

$$\delta J'_2 = \frac{1}{4} \sum_1^2{}_\alpha \left\{ \sum_o^6{}_r\, \delta g_{7-r} \sum_2^4{}_s\, (s-1)\, g_{4-s}\, \frac{h_{r+s,\alpha}}{2L} \right\} J_\alpha$$

$$+ \frac{1}{4} \left\{ -\delta g_2 + \sum_o^6{}_r\, \delta g_{7-r} \sum_2^4{}_s\, (s-1) g_{4-s}\, \frac{h_{r+s,3}}{2L} \right\} J_3$$

$$+ \frac{1}{4} \left\{ \sum_o^6{}_r\, \delta g_{7-r} \sum_2^4{}_s\, (s-1) g_{4-s}\, \frac{h'_{r+s,1}}{2L} \right\} J'_1$$

$$+ \frac{1}{4} \sum_2^3{}_\alpha \left\{ \sum_o^6{}_r\, \delta g_{7-r} \sum_2^4{}_s\, (s-1)\, g_{4-s}\, \frac{h'_{r+s,\alpha}}{2L} - \sum_{3-\alpha}^1{}_s\, (2+s)\, \delta g_{1-s}\, \mathfrak{g}'_{4+s,\alpha} \right\} J'_\alpha$$

$$\delta J'_1 = \frac{1}{4} \sum_o^6{}_r\, \delta g_{7-r} \sum_1^5{}_s\, s g_{5-s}\, \frac{h_{r+s,1}}{2L}\, J_1$$

$$+ \frac{1}{4} \sum_2^3{}_\alpha \Big\{ -(\alpha-1)\, \delta g_{6-\alpha} - \sum_{4-\alpha}^2{}_s\, (3+s)\, \delta g_{2-s}\, \mathfrak{g}_{4+s,\alpha}$$

$$+ \sum_o^6{}_r\, \delta g_{7-r} \sum_1^5{}_s\, s g_{5-s}\, \frac{h_{r+s,\alpha}}{2L} \Big\} J_\alpha$$

$$+ \frac{1}{4} \sum_1^3{}_\alpha \left\{ -\sum_{3-\alpha}^2{}_s\, (3+s)\, \delta g_{2-s}\, \mathfrak{g}'_{4+s,\alpha} + \sum_o^6{}_r\, \delta g_{7-r} \sum_1^5{}_s\, s g_{5-s}\, \frac{h'_{r+s,\alpha}}{2L} \right\} J'_\alpha$$

Vel etiam:

$$\delta J'_2 = \frac{1}{4}\sum_{1}^{2}{}_{\alpha}\left\{\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{2}^{4}{}_{s}\,(s-1)\,g_{4-s}\,\frac{h_{r+s,\alpha}}{2L}\right\}J_\alpha$$

$$+\frac{1}{4}\left\{\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{2}^{4}{}_{s}\,(s-1)\,g_{4-s}\,\frac{h_{r+s,3}}{2L}-\delta g_2\right\}J_3$$

$$+\frac{1}{4}\left\{\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{2}^{4}{}_{s}\,(s-1)\,g_{4-s}\,\frac{h'_{r+s,1}}{2L}\right\}J'_1$$

$$+\frac{1}{4}\left\{\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{2}^{4}{}_{s}\,(s-1)\,g_{4-s}\,\frac{h'_{r+s,2}}{2L}\right\}J'_2$$

$$+\frac{1}{4}\left\{\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{2}^{4}{}_{s}\,(s-1)\,g_{4-s}\,\frac{h'_{r+s,3}}{2L}+2\,\delta g_1\right\}J'_3$$

$$\delta J'_1 = \frac{1}{4}\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\sum_{1}^{5}{}_{s}\,s g_{5-s}\,\frac{h_{r+s,1}}{2L}\,J_1$$

$$+\frac{1}{4}\left\{-\delta g_4+\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{1}^{5}{}_{s}\,s g_{5-s}\,\frac{h_{r+s,2}}{2L}\right\}J_2$$

$$+\frac{1}{4}\left\{-2\,\delta g_3+\frac{g_2}{3}\,\delta g_1+\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{1}^{5}{}_{s}\,s g_{5-s}\,\frac{h_{r+s,3}}{2L}\right\}J_3$$

$$+\frac{1}{4}\left\{\sum_{o}^{6}\,\delta g_{7-r}\ \sum_{1}^{5}{}_{s}\,s g_{5-s}\,\frac{h'_{r+s,1}}{2L}\right\}J'_1$$

$$+\frac{1}{4}\left\{\frac{4}{3}\,\delta g_1+\sum_{o}^{6}\,\delta g_{7-r}\ \sum_{1}^{5}{}_{s}\,s g_{5-s}\,\frac{h'_{r+s,2}}{2L}\right\}J'_2$$

$$+\frac{1}{4}\left\{-\frac{2}{3}\,g_1\,\delta g_1+3\,\delta g_2+\sum_{o}^{6}{}_{r}\,\delta g_{7-r}\ \sum_{1}^{5}{}_{s}\,s g_{5-s}\,\frac{h'_{r+s,3}}{2L}\right\}J'_3$$

Quantitates $h_{r,\alpha}$ sunt hae:

1) $r = 0, 1, 2, \ldots 6$

$$h_{r,1} = -\sum_{1}^{5}{}_{n}\, n g_{5-n} A_{r+n} = -\left\{g_4 A_{r+1} + 2 g_3 A_{r+2} + 3 g_2 A_{r+3} + 4 g_1 A_{r+4} + 20\, A_{r+5}\right\}$$

$$h_{r,2} = A_r\,(g_4 + 5.4\, g_{6,2}) + \sum_{1}^{4}{}_{n}\,(1-n)\, g_{4-n}\, A_{r+n} = -\left\{g_2 A_{r+2} + 2 g_1 A_{r+3} + 12\, A_{r+4}\right\}$$

$$h_{r,3} = \sum_{n=0}^{1} A_{r+n} \left\{ (2-n)\, g_{3-n} + \sum_{m=1}^{2-n} (3+m-n)\, g_{2-m-n}\, \mathfrak{g}_{4+m,3} \right\} + \sum_{n=2}^{3} (2-n) g_3\, A_{r+n}$$

$$= -4\, A_{r+3}$$

$$h'_{r,3} = \sum_{n=0}^{2} A_{r+n} \sum_{m=0}^{2-n} (3+m-n)\, g_{2-m-n}\, \mathfrak{g}'_{4+m,3} = -4\, A_{r+2}$$

$$h'_{r,2} = \sum_{n=0}^{1} A_{r+n} \sum_{m=1}^{2-n} (3+m-n)\, g_{2-m-n}\, \mathfrak{g}'_{4+m,2} = -4\, A_{r+1}$$

$$h'_{r,1} = A_r .\ 5 .\ 4\, \mathfrak{g}'_{6,1} = -4\, A_r$$

2) $\bar{r} = 13 - r > 6$

$$h_{13-r,1} = -\sum_{n}^{1} (n+1)\, g_{6+n}\, A_{12-r-n} = -\left\{ g_6\, A_{12-r} + 2 g_7\, A_{11-r} \right\}$$

$$h_{13-r,2} = -\sum_{n}^{2} A_{12-r-n} \left\{ (2+n)\, g_{5+n} + (6+n)\, g_{1+n}\, \mathfrak{g}_{6,2} \right\}$$

$$- \sum_{n=3}^{5} A_{12-r-n} \left\{ (6+n)\, g_{1+n}\, \mathfrak{g}_{6,2} - A_{6-r}\, 12 g_7\, \mathfrak{g}_{6,2} \right.$$

$$= \frac{1}{20} \left\{ A_{12-r} (6 g_1 g_4 - 40 g_5) + A_{11-r} (7 g_2 g_4 - 60 g_6) + A_{10-r} (8 g_3 g_4 - 80 g_7) \right.$$
$$\left. + 9 g_4^2\, A_{9-r} + 10\, g_4\, g_5\, A_{8-r} + 11 g_4\, g_6\, A_{7-r} + 12\, g_4 g_7\, A_{6-r} \right\}$$

$$h_{13-r,3} = -\sum_{n=0}^{3} A_{12-r-n} \left\{ (3+n) g_{4+n} + \sum_{m=1}^{2} (4+m+n)\, g_{3-m+n}\, \mathfrak{g}_{4+m,3} \right\}$$

$$- A_{8-r} \sum_{m=1}^{2} (8+m)\, g_{7-m}\, \mathfrak{g}_{4+m,3}$$

$$- \sum_{n=5}^{6} A_{12-r-n} \sum_{m=n-4}^{2} (4+m+n)\, g_{3-m+n}\, \mathfrak{g}_{4+m,3}$$

$$= \frac{-1}{60} \left\{ (6 g_1^2 g_2 - 36 g_1 g_3 - 25 g_2^2 + 180 g_4)\, A_{12-r} + (7 g_1 g_2^2 - 72 g_2 g_3 + 240 g_5) A_{11-r} \right.$$
$$+ (8 g_1 g_2 g_3 - 35 g_2 g_4 + 48 g_3^2 + 300 g_6)\, A_{10-r} + (9 g_1 g_2 g_4$$
$$- 40 g_2\, g_5 - 54 g_3\, g_4 + 360 g_7)\, A_{9-r} + (10 g_1 g_2 g_5 - 45 g_2 g_6 - 60 g_3 g_5)\, A_{8-r}$$
$$+ (11 g_1 g_2 g_6 - 50 g_2 g_7 - 66 g_3 g_6)\, A_{7-r} + (12 g_1 g_2 g_7$$
$$\left. - 72 g_3 g_7)\, A_{6-r} \right\}$$

$$h'_{13-r,\,1} = -\sum_{0}^{5} A_{12-r-n}\,(6+n)\,g_{1+n}\,g'_{6,1} - A_{6-r}\,(12 g_7\, g'_{6,1})$$

$$= \frac{1}{5}\left\{6 g_1 A_{12-r} + 7 g_2 A_{11-r} + 8 g_3 A_{10-r} + 9 g_4 A_{9-r} + 10 g_5 A_{8-r} + 11 g_6 A_{7-r} + 12 g_7 A_{6-r}\right\}$$

$$h'_{13-r,\,2} = -\sum_{0}^{4}{}_{n} A_{12-r-n} \sum_{1}^{2}{}_{m} (4+m+n)\,g_{3-m+n}\,g'_{4+m,\,2}$$

$$- \sum_{5}^{6}{}_{n} A_{12-r-n} \sum_{n-4}^{2}{}_{m} (4+m+n)\,g_{3-m+n}\,g'_{4+m,\,2}$$

$$= -\frac{1}{15}\left\{(6 g_1^2 - 25 g_2) A_{12-r} + (7 g_1 g_2 - 30 g_3) A_{11-r} + (8 g_1 g_3 - 35 g_4) A_{10-r} + (9 g_1 g_4 - 40 g_5) A_{9-r} + (10 g_1 g_5 - 45 g_6) A_{8-r} + (11 g_1 g_6 - 50 g_7) A_{7-r} + 12 g_1 g_7 A_{6-r}\right\}$$

$$h'_{13-r,\,3} = -\left\{\sum_{0}^{3} A_{12-r-n} \sum_{0}^{2}{}_{m} (4+m+n)\,g_{3-m+n}\,g'_{4+m,\,3} - \sum_{4}^{6}{}_{n} A_{12-r-n} \sum_{n-4}^{2}{}_{m} (4+m+n)\,g_{3-m+n}\,g'_{4+m,\,3}\right\}$$

$$= \frac{1}{60}\left\{A_{12-r}(12 g_1^3 - 104 g_1 g_2 + 240 g_3) + A_{11-r}(14 g_1^2 g_2 - 60 g_1 g_3 - 63 g_2^2 + 300 g_4) + A_{10-r}(16 g_1^2 g_3 - 70 g_1 g_4 - 72 g_2 g_3 + 360 g_5) + A_{9-r}(18 g_1^2 g_4 - 80 g_1 g_5 - 81 g_2 g_4 + 420 g_6) + A_{8-r}(20 g_1^2 g_5 - 90 g_1 g_6 - 90 g_2 g_5 + 480 g_7) + A_{7-r}(22 g_1^2 g_6 - 100 g_1 g_7 - 99 g_2 g_6) + A_{6-r}(24 g_1^2 g_7 - 108 g_2 g_7)\right\}$$

Ad quantitates $a_{r,s}$ formandas hae inserviunt formulae:

1) $r \leqq s \quad a_{r,s} = (r+1)\,g_{6-r}\,g_{6-s} - (s-r+2)\,g_{7-r}\,g_{5-s} - (s-r+4)\,g_{8-r}\,g_{4-s} - (s-r+6)\,g_{9-r}\,g_{3-s} - (s-r+8)\,g_{10-r}\,g_{2-s} - (s-r+10)\,g_{11-r}\,g_{1-s}$

2) $r > s \quad a_{r,s} = (s+1)\,g_{6-s}\,g_{6-r} - (r-s+2)\,g_{7-s}\,g_{5-r} - \ldots - (r-s+10)\,g_{11-s}\,g_{1-r}$

Tum erit $L = \sum \pm a_{00}\, a_{11} \ldots a_{66} =$

$$= \begin{vmatrix}
g_6^2 - 2 g_7 g_5 & g_6 g_5 - 3 g_7 g_4 & g_6 g_4 - 4 g_7 g_3 & g_6 g_3 - 5 g_7 g_2 & g_6 g_2 - 6 g_7 g_1 & g_6 g_1 - 28 g_7 & 4 g_6 \\
g_6 g_5 - 3 g_7 g_4 & \begin{cases} 2 g_5^2 - 2 g_6 g_4 \\ \quad - 4 g_7 g_3 \end{cases} & \begin{cases} 2 g_5 g_4 - 3 g_6 g_3 \\ \quad 5 g_7 g_2 \end{cases} & \begin{cases} 2 g_5 g_3 - 4 g_6 g_2 \\ \quad - 6 g_7 g_1 \end{cases} & \begin{cases} 2 g_5 g_2 - 5 g_6 g_1 \\ \quad - 28 g_7 \end{cases} & 2 g_5 g_1 - 24 g_6 & 8 g_5 \\
g_6 g_4 - 4 g_7 g_3 & \begin{cases} 2 g_5 g_4 - 3 g_6 g_3 \\ \quad - 5 g_7 g_2 \end{cases} & \begin{cases} 3 g_4^2 - 2 g_5 g_3 \\ - 4 g_6 g_2 - 6 g_7 g_1 \end{cases} & \begin{cases} 3 g_4 g_3 - 3 g_5 g_2 \\ - 5 g_6 g_1 - 28 g_7 \end{cases} & \begin{cases} 3 g_4 g_2 - 4 g_5 g_1 \\ \quad - 24 g_6 \end{cases} & 3 g_4 g_1 - 20 g_5 & 12 g_4 \\
g_6 g_3 - 5 g_7 g_2 & \begin{cases} 2 g_5 g_3 - 4 g_6 g_2 \\ \quad - 6 g_7 g_1 \end{cases} & \begin{cases} 3 g_4 g_3 - 3 g_5 g_2 \\ - 5 g_6 g_1 - 28 g_7 \end{cases} & \begin{cases} 4 g_3^2 - 2 g_4 g_2 \\ - 4 g_5 g_1 - 24 g_6 \end{cases} & \begin{cases} 4 g_3 g_2 - 3 g_4 g_1 \\ \quad - 20 g_5 \end{cases} & 4 g_3 g_1 - 16 g_4 & 16 g_3 \\
g_6 g_2 - 6 g_7 g_1 & \begin{cases} 2 g_5 g_2 - 5 g_6 g_1 \\ \quad - 28 g_7 \end{cases} & \begin{cases} 3 g_4 g_2 - 4 g_5 g_1 \\ \quad - 24 g_6 \end{cases} & \begin{cases} 4 g_3 g_2 - 3 g_4 g_1 \\ \quad - 20 g_5 \end{cases} & \begin{cases} 5 g_2^2 - 2 g_3 g_1 \\ \quad - 16 g_4 \end{cases} & 5 g_2 g_1 - 12 g_3 & 20 g_2 \\
g_6 g_1 - 28 g_7 & 2 g_5 g_1 - 24 g_6 & 3 g_4 g_1 - 20 g_5 & 4 g_3 g_1 - 16 g_4 & 5 g_2 g_1 - 12 g_3 & 6 g_1^2 - 8 g_2 & 24 g_1 \\
4 g_6 & 8 g_5 & 12 g_4 & 16 g_3 & 20 g_2 & 24 g_1 & 112
\end{vmatrix}$$

Tum est $A_{r,s} = A_{r+s} = \dfrac{dL}{d\,a_{r,s}}$

Inter functiones A_α has habemus relationes identicas:

$O = g_7 A_r + g_6 A_{r+1} + g_5 A_{r+2} + g_4 A_{r+3} + g_3 A_{r+4} + g_2 A_{r+5} + g_1 A_{r+6} + 4 A_{r+7}$

vel etiam:

$$O = g_6 A_r + 2 g_5 A_{r+1} + 3 g_4 A_{r+2} + 4 g_3 A_{r+3} + 5 g_2 A_{r+4} + 6 g_1 A_{r+5} + 28 A_{r+6}$$

$$r = 0, 1, 2, 3, 4, 5.$$

Si $g_1 = O$ ponitur, erit

$$y^2 = 4x^7 + g_2 x^5 + g_3 x^4 + g_4 x^3 + g_5 x^2 + g_6 x + g_7$$

et

$$H(xy)_1 = \frac{1}{y} \qquad H'(xy)_1 = -\frac{1}{4}\,\frac{20x^5 + 3g_2 x^3 + 2 g_3 x^2 + g_4 x}{y}$$

$$H(x\,y) = \frac{x}{y} \qquad H'(xy)_2 = -\frac{1}{4}\,\frac{20x^4 + g_2 x^2}{}$$

$$H(xy)_3 = \frac{x^2}{y} \qquad H'(xy)_3 = -\frac{x^3}{y}$$

Tum nascetur

$$h_{r,1} = -(g_4 A_{r+1} + 2 g_3 A_{r+2} + 3 g_2 A_{r+3} + 20 A_{r+5}), \quad h'_{r,1} = -4 A_r$$

$$h_{r,2} = -(g_2 A_{r+2} + 12 A_{r+4}) \qquad h'_{r,2} = -4 A_{r+1}$$

$$h_{r,3} = -4 A_{r+3} \qquad h'_{r,3} = -4 A_{r+2}$$

$$(r = 0, 1, 2, 3, 4, 5, 6.)$$

$$h_{13-r,1} = -(g_6 A_{12-r} + 2 g_7 A_{11-r})$$

$$h_{13-r,2} = \frac{1}{20}\left\{-40 g_5 A_{12-r} + (7 g_2 g_4 - 60 g_6) A_{11-r} + (8 g_3 g_4 - 80 g_7) A_{10-r} + 9 g_4^2 A_{9-r} + 10 g_4 g_5 A_{8-r} + 11 g_4 g_6 A_{7-r} + 12 g_4 g_7 A_{6-r}\right\}$$

$$h_{13-r,3} = \frac{1}{60}\left\{(25 g_2^2 - 180 g_4) A_{12-r} + (72 g_2 g_3 - 240 g_5) A_{11-r} + (35 g_2 g_4 + 48 g_3^2 - 300 g_6) A_{10-r} + (40 g_2 g_5 + 54 g_3 g_4 - 360 g_7) A_{9-r} + (45 g_2 g_6 + 60 g_3 g_5) A_{8-r} + (50 g_2 g_7 + 66 g_3 g_6) A_{7-r} + 72 g_3 g_7 A_{6-r}\right\}$$

$$h'_{13-r,1} = \frac{1}{5}\left\{7 g_2 A_{11-r} + 8 g_3 A_{10-r} + 9 g_4 A_{9-r} + 10 g_5 A_{8-r} + 11 g_6 A_{7-r} + 12 g_7 A_{6-r}\right\}$$

$$h'_{13-r,2} = \frac{1}{15}\left\{25 g_2 A_{12-r} + 30 g_3 A_{11-r} + 35 g_4 A_{10-r} + 40 g_5 A_{9-r} + 45 g_6 A_{8-r} + 50 g_7 A_{7-r}\right\}$$

$$h'_{13-r,3} = \frac{1}{20}\left\{80 g_3 A_{12-r} + (100 g_4 - 21 g_2^2) A_{11-r} + (120 g_5 - 24 g_2 g_3) A_{10-r} + (140 g_6 - 27 g_2 g_4) A_{9-r} + (160 g_7 - 30 g_2 g_5) A_{8-r} - 33 g_2 g_6 A_{7-r} - 36 g_2 g_7 A_{6-r}\right\}$$

$$(r = 2, 3, 4.)$$

Aequationes differentiales has induunt formas:

$$\delta J\beta = \sum_{1}^{5}{}_{\alpha} \left\{ \sum_{o}^{5}{}_{r} \frac{-h_{r+\beta-1,\alpha}}{2L} \delta g_{7-r} \; J_\alpha + \sum_{o}^{5}{}_{r} \frac{-h'_{r+\beta-1,\alpha}}{2L} \delta g_{7-r} J'_\alpha \right\}$$

$$(\beta = 1, 2, 3)$$

$$\delta J'_1 = \frac{1}{4} \left\{ \sum_{o}^{5}{}_{r} \delta g_{7-r} \sum_{1}^{5}{}'_{s} \frac{s g_{5-s} \, h_{r+s,1}}{2L} \right\} J_1$$

$$+ \frac{1}{4} \left\{ \sum_{o}^{5}{}_{r} \delta g_{7-r} \sum_{1}^{5}{}'_{s} \frac{s g_{5-s} \, h_{r+s,2}}{2L} - \delta g_4 \right\} J_2$$

$$+ \frac{1}{4} \left\{ \sum_{o}^{5}{}_{r} \delta g_{7-r} \sum_{1}^{5}{}'_{s} \frac{s g_{5-s} \, h_{r+s,3}}{2L} - 2\delta g_3 \right\} J_3$$

$$+ \frac{1}{4} \sum_{o}^{5} \delta g_{7-r} \sum_{1}^{5}{}'_{s} \frac{s g_{5-s} \, h'_{r+s,1}}{2L} \Big\} J'_1$$

$$+ \frac{1}{4} \left\{ \sum_{o}^{6}{}_{r} \delta g_{7-r} \sum_{1}^{5}{}' \frac{s g_{5-s} \, h'_{r+s,2}}{2L} \right\} J'_2$$

$$+ \frac{1}{4} \left\{ \sum_{o}^{5} \delta g_{7-r} \sum_{1}^{5}{}'_{s} \frac{s g_{5-s} \, h'_{r+s,3}}{2L} + 3\delta g_2 \right\} J'_3$$

$$\delta J'_2 = \frac{1}{4} \sum_{1}^{2}{}_{\alpha} \left\{ \sum_{o}^{5} \delta g_{7-r} \sum_{2}^{4}{}'_{s} \frac{(s-1) g_{4-s} \, h_{r+s,\alpha}}{2L} \right\} J_\alpha$$

$$+ \frac{1}{4} \left\{ \sum_{o}^{5}{}_{r} \delta g_{7-r} \sum_{2}^{4}{}'_{s} \frac{(s-1) g_{4-s}, \, h_{r+s,3}}{2L} - \delta g_2 \right\} J_3$$

$$+ \frac{1}{4} \sum_{1}^{3}{}'_{\alpha} \left\{ \sum_{o}^{5} \delta g_{7-r} \sum_{2}^{4}{}'_{s} \frac{(s-1) g_{4-s} \, h'_{r+s,\alpha}}{2L} \right\} J'_\alpha$$

$$\delta J'_3 = \frac{1}{4} \sum_{1}^{3}{}_{\alpha} \left\{ \sum_{o}^{5} \frac{h_{r+3,\alpha}}{2L} \delta g_{7-r} \; J_\alpha + \sum_{o}^{5}{}_{r} \frac{h'_{r+3,\alpha}}{2L} \delta g_{7-r} \; J'_\alpha \right\}$$

Signo summationis $\sum$ secundum s additum est „′", ne obliviscaris omittendum esse membrum, ubicunque g_1 reperitur; g_o autem $= 4$ est ponendum.

B.
Methodus altera.

Inducantur nunc pro variationibus modulorum variationes radicum aequationis $R(x) = O.$ Sit

$$1)\qquad y^2 = R(x) = 4(x-e_1)(x-e_2)\dots(x-e_{2\varrho+1}) = 4\prod_{\nu=1}^{2\varrho+1}(x-e_\nu)$$

Functiones $H(xy)_\alpha$ et $H'(xy)_\alpha$ in eadem forma supponuntur:

$$H(xy)_\alpha = \frac{x^{\alpha-1}}{y},$$

$$H'(xy)_\alpha = -\frac{1}{4}\,\frac{(2\rho-2\alpha+1)4x^{2\varrho-\alpha}+(2\rho-2\alpha)g_1x^{2\varrho-\alpha-1}+\dots+g_{2\varrho-2\alpha}x^\alpha}{y} = \frac{H(x)_\alpha}{y}$$

$$\alpha = 1, 2, 3, \dots \rho$$

Jam et hic de hujus formae integrali:

$$\int \frac{x^n\,dx}{\sqrt{Rx}}$$

primum tractemus. Erit:

$$\delta\int\frac{x^n\,dx}{\sqrt{Rx}} = \int x^n\,dx\,\delta\left(\frac{1}{\sqrt{Rx}}\right) = \int\frac{-x^n\,dx\,\delta Rx}{2Rx\sqrt{Rx}}$$

est autem $\delta Rx = \delta\prod_{1}^{2\varrho+1}(x-e_\nu) = \sum_{\mu=1}^{2\varrho+1}\frac{-\delta e_\mu\prod_{1}^{2\varrho+1}(x-e_\nu)}{x-e_\mu} = -Rx\sum_{\mu=1}^{2\varrho+1}\frac{\delta e_\mu}{x-e_\mu}$

habemus ergo:

$$2)\qquad \delta\int\frac{x^n\,dx}{\sqrt{Rx}} = \int\frac{-x^n\,dx\left(-Rx\sum_{\mu=1}^{2\varrho+1}\frac{\delta e_\mu}{x-e_\mu}\right)}{2Rx\sqrt{Rx}}$$

$$= \int\sum_{\mu=1}^{2\varrho+1}\frac{\delta e_\mu\,x^n}{2(x-e_\mu)}\,\frac{dx}{\sqrt{Rx}} = \sum_{\mu=1}^{2\varrho+1}\delta e_\mu\int\frac{x^n}{2(x-e_\mu)}\,\frac{dx}{\sqrt{Rx}}$$

Quum autem pro $\frac{x^n}{x-e_\mu}$ scribi possit $\frac{x^n-e_\mu{}^n}{x-e_\mu}+\frac{e_\mu{}^n}{x-e_\mu} =$

$$x^{n-1}+x^{n-2}e_\mu+\dots+e_\mu{}^{n-1}+\frac{e_\mu{}^n}{x-e_\mu},$$

fiet

$$3)\qquad \delta\int\frac{x^n\,dx}{\sqrt{Rx}}=\sum_{1}^{2\varrho+1}{}_{\mu}\ \delta e_\mu\int\frac{x^{n-1}+e_\mu\,x^{n-2}+\ldots+e_\mu{}^{n-1}}{\sqrt{Rx}}\,dx$$
$$+\sum_{1}^{2\varrho+1}{}_{\mu}\ e_\mu{}^n\,\delta e_\mu\int\frac{1}{2(x-e_\mu)}\frac{dx}{\sqrt{Rx}}$$

Pars prior facile ad integralia nostra primae ac secundae speciei revocatur; similiter sese cum altera parte res habet, quum in nulla evolutione reperiatur $\int(x-e_\mu)^{-1}\,dx$, quod facillime intelligi potest. Qua autem ratione hoc integrale reducendum sit, nunc quaeramus.

Proficiscimur ab illa formula cardinali:

$$\frac{d}{dx}H(x\,y,x'\,y')-\frac{d}{dx'}H(x'\,y'\,x\,y)=\sum_{1}^{\varrho}{}_{\alpha}\,(H(x'y')_\alpha\ \ H'(xy)_\alpha-H(xy)_\alpha\ H'(x'y')_\alpha)$$

quam, hanc formam induere perspectum habemus (A, 3):

$$\frac{d}{dx}\frac{y}{2(x'-x)y'}-\frac{d}{dx'}\frac{y'}{2(x-x')y}=\sum_{1}^{\varrho}{}_{\alpha}\left(\frac{x'^{\alpha-1}}{y'}\frac{H(x)_\alpha}{y}-\frac{x^{\alpha-1}}{y}\frac{\mathrm{H}(x')_\alpha}{y'}\right)$$

vel

$$4)\quad -d\frac{y'}{2(x-x')y}=-\frac{d}{dx}\frac{y\,dx'}{2(x'-x)y'}+\sum_{1}^{\varrho}{}_{\alpha}\left(\frac{x'^{\alpha-1}}{y'}\frac{\mathrm{H}(x)_\alpha}{y}-\frac{x^{\alpha-1}}{y}\frac{\mathrm{H}(x')_\alpha}{y'}\right)dx'$$

Ponamus

$$\frac{dx'}{2(x'-x)y'}=d\,(y'\,P(x'|e_\mu,x))$$

$$5)\qquad \frac{x'^{\alpha-1}\,dx'}{y'}=-\,d\,(y'\,P_\alpha\,(x'|e_\mu))$$

$$\frac{\mathrm{H}(x')_\alpha\,dx'}{y'}=d\,(y'\,\overline{P}_\alpha\,(x'|e_\mu))$$

ubi P, P_α, $\overline{P}_\alpha$ series sunt infinitae; habebimus, integratione facta:

$$-\frac{y'}{2(x-x')y}=-\frac{d}{dx}\,y\,y'\,P(x'|e_\mu,x)$$
$$+\sum_{1}^{\varrho}{}_{\alpha}\left\{y'\,P_\alpha\,(x'|e_\mu)\,\frac{\mathrm{H}'(x)_\alpha}{y}-y'\,\overline{P}_\alpha\,(x'|e_\mu)\,\frac{x^{\alpha-1}}{y}\right\}$$

Constans non additur, quum pro $x'=e_\mu$ utraque pars evanescat. ($y'=o$). Omisso factore communi y' et signis mutatis orietur aequatio:

$$6)\qquad \frac{1}{2\,(x-x')y} = \frac{d}{dx}\, y\, P\,(x'|\,e_\mu, x) + \sum_1^{\varrho}{}_\alpha \left\{ \frac{x^{\alpha-1}}{y}\, \overline{P}_\alpha\,(x'|e_\mu) - \frac{H(x)_\alpha}{y}\, P_\alpha\,(x'|e_\mu) \right\}$$

Evolvatur pars sinistra secundum positivas potestates argumenti $(x'-e_\mu)$; erit:

$$\frac{1}{2\,(x-x')} = \frac{1}{2\,(x-e_\mu)\,(1-\frac{x'-e_\mu}{x-e_\mu})y} = \frac{1}{2y}\left\{\frac{1}{x-e_\mu} + \frac{x'-e_\mu}{(x-e_\mu)^2} + \dots\right\}$$

Quum autem sint, (quod statim demonstrabitur):

$$\begin{aligned} &P\,(x'|e_\mu, x) = p\,(x|e_\mu)_0 + p\,(x|e_\mu)_1\,(x'-e_\mu) + \dots \\ 7)\qquad &P_\alpha\,(x'|e_\mu) = p\,(e_\mu)_{\alpha,0} + p\,(e_\mu)_{\alpha,1}\,(x'-e_\mu) + \dots \\ &\overline{P}_\alpha\,(x'|e_\mu) = \overline{p}\,(e_\mu)_{\alpha,0} + \overline{p}\,(e_\mu)_{\alpha,1}\,(x'-e_\mu) + \dots \end{aligned}$$

erit aequatio nostra:

$$\begin{aligned} 8)\qquad \frac{1}{2y\,(x-e_\mu)} + \frac{x'-e_\mu}{2y\,(x-e_\mu)^2} + \dots = {}& \frac{d}{dx}\, yp(x|e_\mu)_0 + (x'-e_\mu)\frac{d}{dx}\, y\, p\,(x|e_\mu)_1 + \dots \\ & + \sum_1^{\varrho}{}_\alpha \left\{ \frac{x^{\alpha-1}}{y}\left(\overline{p}\,(e_\mu)_{\alpha,0} + (x'-e_\mu)\,p\,(e_\mu)_{\alpha,1} + \dots\right) \right. \\ & \left. - \frac{H(x)_\alpha}{y}\left(p\,(e_\mu)_{\alpha,0} + (x'-e_\mu)\,p\,(e_\mu)_{\alpha,1} + \dots\right) \right\} \end{aligned}$$

Quum aequatio haec identica sit, coëfficientes singularium potestatum $(x'-e_\mu)$ iidem esse debent in utraque parte. Comparemus hoc modo membra constantia; habebimus:

$$9)\qquad \frac{1}{2\,y\,(x-e_\mu)} = \frac{d}{dx}\, y\, p\,(x|e_\mu)_0 + \sum_1^{\varrho}{}_\alpha \left(\overline{p}_{\alpha,0}\, \frac{x^{\alpha-1}}{y} - p_{\alpha,0}\,\frac{H(x)_\alpha}{y}\right)$$

F o r m u l a h a e c c o n t i n e t r e d u c t i o n e m q u a e s i t a m :

$$\begin{aligned} \int \frac{dx}{2\,y\,(x-e_\mu)} = \int \frac{dx}{2\,(x-e_\mu)\,\sqrt{Rx}} &= \sum_1^{\varrho}{}_\alpha \left(\overline{p}\,(e_\mu)_{\alpha,0} \int \frac{x^{\alpha-1}\,dx}{y} - p\,(e_\mu)_{\alpha,0} \int \frac{H(x)_\alpha}{y}\,dx\right) \\ &\qquad + yp\,(x|e_\mu)_0 \\ &= \sum_1^{\varrho}{}_\alpha \left(\overline{p}\,(e_\mu)_{\alpha,0} \int H\,(xy)_\alpha\, dx - p\,(e_\mu)_{\alpha,0} \int H'\,(xy)_\alpha\, dx\right) + \frac{R(x)\, p\,(x|e_\mu)_0}{\sqrt{R(x)}} \end{aligned}$$

et

$$\begin{aligned} 10)\qquad \overline{\int} \frac{dx}{2\,(x-e_\mu)\,\sqrt{Rx}} &= \sum_1^{\varrho}{}_\alpha \left(\overline{p}\,(e_\mu)_{\alpha,0} \overline{\int} H\,(xy)_\alpha\, dx - p\,(e_\mu)_{\alpha,0} \overline{\int} H'\,(xy)_\alpha\, dx\right) \\ &= \sum_1^{\varrho}{}_\alpha \left(\overline{p}\,(e_\mu)_{\alpha,0} J_\alpha - p\,(e_\mu)_{\alpha,0} J'_\alpha\right) \end{aligned}$$

Restat, ut demonstremus, nostras series P talem formam exhibere, quam supposuimus. (Cf. 7.)

Posuimus autem primum:

$$\frac{dx'}{2(x'-x)y'} = d(y' P(x'|e_\mu x)) = d(y' \{p(x|e_\mu)_o + p(x|e_\mu)_1 (x'-e_\mu) + \ldots\})$$

Jam has habemus evolutiones:

$$\frac{1}{y'} = \frac{1}{\sqrt{4(x'-e_1)(x-e_2)\ldots(x'-e_{2\varrho+1})}} = \frac{1}{\sqrt{x'-e_\mu}\sqrt{R' e_\mu}}\left\{1 + (x'-e_\mu)\sum_{\nu \gtrless \mu}' \frac{1}{2(e_\nu - e_\mu)} + \ldots\right\}$$

$$\frac{1}{x'-x} = \frac{-1}{x-e_\mu}\left\{1 + \frac{x'-e_\mu}{x-e_\mu} + \ldots\right\}$$

ergo:

$$\frac{1}{2(x'-x)y'} = \frac{-1}{2(x-e_\mu)\sqrt{x'-e_\mu}\sqrt{R'(e_\mu)}}\left\{1 + (x'-e_\mu)\left(\sum_{\nu \gtrless \mu}' \frac{1}{2(e_\nu - e_\mu)} + \frac{1}{x-e_\mu}\right) + \ldots\right\}$$

$$= \frac{-1}{2(x-e_\mu)\sqrt{R'(e_\mu)}}\left\{(x'-e_\mu)^{-\frac{1}{2}} + (x'-e_\mu)^{\frac{1}{2}}\mathfrak{P}_o(x'-e_\mu)\right\}$$

$$= \frac{1}{2}\frac{d}{dx'}\frac{-1}{(x-e_\mu)\sqrt{R' e_\mu}}\left\{2(x'-e_\mu)^{\frac{1}{2}} + (x'-e_\mu)^{\frac{3}{2}}\overline{\mathfrak{P}}_o(x'-e_\mu)\right\}$$

$$= -\frac{d}{dx'}\frac{\sqrt{x'-e_\mu}}{(x-e_\mu)\sqrt{R' e_\mu}}\left\{1 + (x'-e_\mu)\mathfrak{P}(x'-e_\mu)\right\}$$

Quum autem sit:

$$\frac{1}{y'} = \frac{1}{\sqrt{R' e_\mu}\sqrt{x'-e_\mu}}\left\{1 + (x'-e_\mu)\sum' \frac{1}{2(e_\nu - e_\mu)} + \ldots\right\}$$

licet scribere:

$$\sqrt{R' e_\mu} = \frac{y'}{\sqrt{x'-e_\mu}}\left\{1 + (x'-e_\mu)\mathfrak{P}_1(x'-e_\mu)\right\}$$

vel etiam

$$\frac{1}{\sqrt{R' e_\mu}} = \frac{y'}{\sqrt{x'-e_\mu}\, R'(e_\mu)}\left\{1 + (x'-e_\mu)\mathfrak{P}_1(x'-e_\mu)\right\}$$

et

$$\frac{\sqrt{x'-e_\mu}}{\sqrt{R' e_\mu}} = \frac{y'}{R'(e_\mu)}\left\{1 + (x'-e_\mu)\mathfrak{P}_1(x'-e_\mu)\right\}$$

Quod si hic valor pro $\frac{\sqrt{x'-e_\mu}}{\sqrt{R' e_\mu}}$ in formula nostra substituitur, erit

$$\frac{1}{2(x'-x)y'} = -\frac{d}{dx'}\frac{y'}{(x-e_\mu)R' e_\mu}\left\{1 + (x'-e_\mu)\mathfrak{P}_2(x'-e_\mu)\right\}$$

$$= \frac{d}{dx}\, y' \left\{\frac{-1}{(x-e_\mu)R'(e_\mu)} + (x'-e_\mu)\mathfrak{P}_3(x'-e_\mu)\right\} = \frac{d}{dx}(y' P(x'|x, e_\mu))$$

Habemus igitur seriem quaesitam $P(x'|x, e_\mu)$, quae integras et positivas tantum continet potestates argumenti $(x'-e_\mu)$. Primus coefficiens est:

11) $$p(x|e_\mu) = -\frac{1}{(x-e_\mu)\; R'(e_\mu)}$$

Simili modo erit

$$\frac{x'^{\alpha-1}}{y'} = \frac{\left(e_\mu + (x_\mu - e_\mu)\right)^{\alpha-1}}{y'}$$

$$= \frac{e_\mu^{\alpha-1} + (\alpha-1)(x_\mu - e_\mu) + \ldots + (x-e_\mu)^{\alpha-1}}{\sqrt{R'(e_\mu)}\;\sqrt{x'-e_\mu}} \left\{1 + (x'-e_\mu)\,\mathfrak{P}_1\,(x'-e_\mu)\right\}$$

$$= \frac{1}{\sqrt{R'(e_\mu)}} \left\{\frac{e_\mu^{\alpha-1}}{\sqrt{x'-e_\mu}} + \sqrt{x'-e_\mu}\;\mathfrak{P}_4\,(x'-e_\mu)\right\}$$

$$= \frac{d}{dx'}\frac{1}{\sqrt{R'e_\mu}} \left\{2\,e_\mu^{\alpha-1}\;\sqrt{x'-e_\mu} + \sqrt{(x'-e_\mu)^3}\;\mathfrak{P}_5\,(x'-e_\mu)\right\}$$

$$= \frac{d}{d\,x'}\frac{\sqrt{x'-e_\mu}}{\sqrt{R'e_\mu}} \left\{2\,e_\mu^{\alpha-1} + (x'-e_\mu)\,\mathfrak{P}_5\,(x'-e_\mu)\right\}$$

$$= \frac{d}{d\,x'}\frac{y'}{R'(e_\mu)} \left\{2\,e_\mu^{\alpha-1} + (x'-e_\mu)\;\mathfrak{P}_6\,(x'-e_\mu)\right\} = \frac{d}{d\,x'}\left(y'\,P_\alpha\;(x'|e_\mu)\right)$$

12) $$p(e_\mu)_{\alpha,\delta} = \frac{2\,e_\mu^{\alpha-1}}{R'(e_\mu)}$$

Si nunc similem evolutionem pro $\dfrac{H'(x)_\alpha}{y}$ quaeremus, habebimus primum:

$$H(x')_\alpha = H(e_\mu)_\alpha + (x'-e_\mu)\,H'(e_\mu)_\alpha + \ldots$$

et

$$\frac{1}{y'} = \frac{1}{\sqrt{R'\,e_\mu}\;\;\sqrt{x'-e_\mu}}\left\{1 + \mathfrak{P}(x'-e_\mu)\right\}$$

ergo:

$$\frac{H(x')_\alpha}{y'} = \frac{1}{\sqrt{R'e_\mu}}\left\{\frac{H\,(e_\mu)_\alpha}{\sqrt{x'-e_\mu}} + \sqrt{x'-e_\mu}\;\mathfrak{P}_7\,(x'-e_\mu)\right\}$$

$$= \frac{d}{dx'}\frac{1}{\sqrt{R'e_\mu}}\left\{2\,H(e_\mu)\;\sqrt{x'-e_\mu} + \sqrt{(x'-e_\mu)^3}\;\mathfrak{P}_8(x-e_\mu)\right\}$$

$$= \frac{d}{dx'}\left\{y'\left(\frac{2\,H(e_\mu)_\alpha}{R'\,e_\alpha}\right) + (x-e_\mu)\;\mathfrak{P}_9\,(x-e_\mu)\right\} = \frac{d}{dx'}\left(y'\,P_\alpha\,(x'|e_\mu)\right)$$

13) et $$p'\,(e_\mu)_{\alpha,\delta} = \frac{2\,H(e_\mu)_\alpha}{R'\,(e_\mu)}$$

Substitutis his valoribus in nostra reductionis formula 10), existet:

$$+\sum_{1}^{\varrho}{}_{\alpha}\int \overline{H\,(x\,y)_\alpha\,dx}\sum_{1}^{2\varrho+1}{}_{\mu}\frac{2\,e_\mu{}^{\varrho+m}\,H\,(e_\mu)}{R'\,(e_\mu)}\,\delta e_\mu$$

$$-\sum_{1}^{\varrho}{}_{\alpha}\int \overline{H'\,(xy)_\alpha\,dx}\sum_{1}^{2\varrho+1}{}_{\mu}\,2\,e_\mu{}^{\varrho+m+\alpha-1}\,\frac{\delta e_\mu}{R'(e_\mu)}$$

Habemus autem

$$\sum_{1}^{m-1}{}_{t}\;\sum_{1}^{2\varrho+1}{}_{\mu}\frac{1}{2}\,e_\mu{}^{m-t-1}\,\delta e_\mu\sum_{\varrho-t+1}^{\varrho}{}_{t}\,\mathfrak{g}_{\varrho+t+1,\alpha}\int \overline{H\,(xy)_\alpha\,dx}$$

$$=\sum_{\varrho-m+2}^{\varrho}{}_{\alpha}\int \overline{H\,(xy)_\alpha\,dx}\sum_{\varrho-\alpha+1}^{m-1}{}_{t}\,\mathfrak{g}_{\varrho+t+1,\alpha}\sum_{1}^{2\varrho+1}{}_{\mu}\frac{1}{2}\,e_\mu{}^{m-t-1}\,\delta e_\mu$$

et etiam, si pro $-\frac{1}{2}\,e_\mu{}^{m-1}\int\overline{H'(xy)_\varrho\,dx}$ ponitur $\frac{1}{2}\,\mathfrak{g}'_{\varrho+1,\varrho}\,e_\mu{}^{m-1}\int\overline{H'(xy)_\varrho\,dx}$

$$\sum_{o}^{m-1}{}_{l}\;\sum_{1}^{2\varrho+1}{}_{\mu}\frac{1}{2}\,e_\mu{}^{m-t-1}\,\delta e_\mu\sum_{\varrho-t}^{\varrho}{}_{\alpha}\mathfrak{g}'_{\varrho+t+1,\alpha}\int\overline{H'\,(xy)_\alpha\,dx}$$

$$=\sum_{\varrho-m+1}^{\varrho}{}_{\alpha}\int\overline{H'\,(xy)_\alpha\,dx}\sum_{\varrho-\alpha}^{m-1}{}_{t}\,\mathfrak{g}'_{\varrho+t+1,\alpha}\sum_{1}^{2\varrho+1}{}_{\mu}\frac{1}{2}\,e_\mu{}^{m-t-1}\,\delta e_\mu$$

Est igitur formula nostra:

$$\delta\int\overline{\frac{x^{\varrho+m}\,dx}{\sqrt{Rx}}}=\sum_{1}^{\varrho}{}_{\alpha}J_\alpha\sum_{1}^{2\varrho+1}{}_{\mu}\left(\frac{e_\mu{}^{\varrho+m-\alpha}}{2}+\frac{2\,e_\mu{}^{\varrho+m}\,H\,(e_\mu)_\alpha}{R'\,(e_\mu)}\right)\delta e_\mu$$

$$+\sum_{\varrho-m+2}^{\varrho}{}_{\alpha}J_\alpha\sum_{\varrho-\alpha+1}^{m-1}{}_{t}\,\mathfrak{g}_{\varrho+t+1,\alpha}\sum_{1}^{2\varrho+1}{}_{\mu}\frac{1}{2}\,e_\mu{}^{m-t-1}\,\delta e_\mu$$

$$-\sum_{1}^{\varrho}{}_{\alpha}J'_\alpha\sum_{1}^{2\varrho+1}\frac{2\,e_\mu{}^{\varrho+m+\alpha-1}}{R'\,(e_\mu)}\,\delta e_\mu$$

$$+\sum_{\varrho-m+1}^{\varrho}{}_{\alpha}J'_\alpha\sum_{\varrho-\alpha}^{m-1}{}_{t}\,\mathfrak{g}'_{\varrho+t+1,\alpha}\sum_{1}^{2\varrho+1}{}_{\mu}\frac{1}{2}\,e_\mu{}^{m-t-1}\,\delta e_\mu$$

16) $$=\sum_{1}^{\varrho-m+1}{}_{\alpha}J_\alpha\sum_{1}^{2\varrho+1}{}_{\mu}\left\{\frac{2\,e_\mu{}^{\varrho+m}\,H\,(e_\mu)_\alpha}{R'(e_\mu)}+\frac{1}{2}\,e_\mu{}^{\varrho+m-\alpha}\right\}\delta e_\mu$$

$$+\sum_{\varrho-m+2}^{\varrho}{}_{\alpha}J_\alpha\sum_{1}^{2\varrho+1}{}_{\mu}\left(\frac{e_\mu{}^{\varrho+m-\alpha}}{2}+\frac{2\,e_\mu{}^{\varrho+m}\,H(e_\mu)_\alpha}{R'\,(e_\mu)}+\sum_{\varrho-\alpha+1}^{m-1}{}_{t}\,\mathfrak{g}_{\varrho+t+1,\alpha}\,\frac{e_\mu{}^{m-t-1}}{2}\right)\delta e_\mu\;-$$

$$- \sum_{1}^{\varrho-m}{}_{\alpha} J'_\alpha \sum_{1}^{2\varrho+1} \frac{2 e_\mu{}^{\varrho+m+\alpha-1}}{R'(e_\mu)} \delta e_\mu$$

$$+ \sum_{\varrho-m+1}^{\varrho}{}_{\alpha} J'_\alpha \sum_{1}^{2\varrho+1}{}_{\mu} \left(\sum_{\varrho-\alpha}^{m-1}{}_{t} \frac{1}{2} \mathfrak{g}'_{\varrho+t+1,\alpha}\, e_\mu{}^{m-t-1} - \frac{2 e_\mu{}^{\varrho+m+\alpha-1}}{R'(e_\mu)} \right) \delta e_\mu$$

$$= \sum_{1}^{\varrho} \mathfrak{h}_{\varrho+m,\alpha} J_\alpha + \sum_{1}^{\varrho} \mathfrak{h}'_{\varrho+m,\alpha} J'_\alpha$$

Scripsimus brevitatis causa:

$$\delta \int \frac{x^n\, dx}{y} = \sum_{1}^{\varrho} \mathfrak{h}_{n,\alpha} J_\alpha + \sum_{1}^{\varrho} \mathfrak{h}'_{n,\alpha} J'_\alpha$$

habemus:

I) si $n \leqq \rho$

17)

$$\mathfrak{h}_{n,\alpha} = \sum_{1}^{2\varrho+1}{}_{\mu} \left\{ \frac{1}{2} e_\mu{}^{n-\alpha} + \frac{2 e_\mu{}^n H(e_\mu)_\alpha}{R'(e_\mu)} \right\} \delta e_\mu \qquad \alpha \leqq n$$

$$\mathfrak{h}_{n,\alpha} = \sum_{1}^{2\varrho+1}{}_{\mu} \left\{ \frac{2 e_\mu{}^n H(e_\mu)_\alpha}{(R'(e_\mu)} \right\} \delta e_\mu \qquad \alpha > n$$

$$\mathfrak{h}'_{n,\alpha} = - \sum_{1}^{2\varrho+1}{}_{\mu} \frac{2 e_\mu{}^{n+\alpha-1}}{R'(e_\mu)} \delta e_\mu \qquad \text{pro omni } \alpha$$

II) $n > \rho \quad n = \rho + m$

17a)

$$\mathfrak{h}_{\varrho+m,\alpha} = \sum_{1}^{2\varrho+1}{}_{\mu} \left(\frac{1}{2} e_\mu{}^{\varrho+m-\alpha} + \frac{2 e_\mu{}^{\varrho+m} H(e_\mu)_\alpha}{R'(e_\mu)} \right) \delta e_\mu \qquad \alpha = 1, 2, .\,, \rho - m + 1$$

$$\mathfrak{h}_{\varrho+m,\alpha} = \sum_{1}^{2\varrho+1}{}_{\mu} \left(\frac{1}{2} e_\mu{}^{\varrho+m-\alpha} + \frac{2 e_\mu{}^{\varrho+m} H(e_\mu)_\alpha}{R'(e_\mu)} + \sum_{\varrho-\alpha+1}^{m-1}{}_{t} \mathfrak{g}_{\varrho+t+1,\alpha} \frac{e_\mu{}^{m-t-1}}{2} \right) \delta e_\mu \qquad \alpha = \rho - m + 2, \ldots \rho$$

$$\mathfrak{h}'_{\varrho+m,\alpha} = - \sum_{1}^{2\varrho+1}{}_{\mu} \frac{2 e_\mu{}^{\varrho+m+\alpha-1} \delta e_\mu}{R' e_\mu} \qquad \alpha = 1, 2, \ldots, \rho - m$$

$$\mathfrak{h}'_{\varrho+m,\alpha} = \sum_{1}^{2\varrho+1}{}_{\mu} \left\{ \sum_{\varrho-\alpha}^{m-1}{}_{t} \frac{1}{2} \mathfrak{g}'_{\varrho+t+1,\alpha}\, e_\mu{}^{m-t-1} - \frac{2 e_\mu{}^{\varrho+m+\alpha-1}}{R'(e_\mu)} \right\} \delta e_\mu$$

$$\alpha = \rho - m + 1, \ldots, \rho$$

Positum est

$$H(xy)^n = \frac{x^{n-1}}{y}$$

erit ergo:

$$18)\qquad \delta\int\overline{H}(xy)_n\,dx = \delta J_n = \delta\int\overline{\frac{x^{n-1}}{y}}\,dx = \sum_{1}^{\varrho}{}_\alpha\left(\mathfrak{h}_{n-1,\alpha}J_\alpha + \mathfrak{h}'_{n-1,\alpha}J'_\alpha\right)\quad n \leqq \rho$$

$$19)\qquad \delta\int\overline{H'}(xy)_\varrho\,dx = -\delta\int\overline{\frac{x^\varrho\,dx}{y}} = -\sum_{1}^{\varrho}{}_\alpha\left(\mathfrak{h}_{\varrho,\alpha}J_\alpha + \mathfrak{h}'_{\varrho,\alpha}J'_\alpha\right) = \delta J'_\varrho$$

Tum autem est

$$H'(xy)_\alpha = -\frac{1}{4}\sum_{\alpha}^{2\varrho-\alpha}{}_s(s-\alpha+1)g_{2\varrho-\alpha-s}\frac{x^s}{y}\qquad (\alpha = 1,2,\ldots\rho-1)$$

$$= -\frac{1}{4}\sum_{\alpha}^{\varrho}{}_s(s-\alpha+1)\,g_{2\varrho-\alpha-s}\frac{x^s}{y}$$

$$-\frac{1}{4}\sum_{1}^{\varrho-\alpha}{}_s(\rho+s-\alpha+1)\,g_{\varrho-\alpha-s}\frac{x^{\varrho+s}}{y}$$

$$\delta\int\overline{H'}(xy)_\alpha\,dx = -\frac{1}{4}\sum_{\alpha}^{\varrho}{}_s(s-\alpha+1)\,\delta g_{2\varrho-\alpha-s}\int\frac{x^s}{y}\,dx$$

$$-\frac{1}{4}\sum_{1}^{\varrho-\alpha}{}_s(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\int\frac{x^{\varrho+s}}{y}\,dx$$

$$-\frac{1}{4}\sum_{\alpha}^{2\varrho-\alpha}{}_s(s-\alpha+1)g_{2\varrho-\alpha-s}\,\delta\int\overline{\frac{x^s\,dx}{y}}$$

$$\delta\int\overline{H'}(xy)_\alpha\,dx = -\frac{1}{4}\sum_{\alpha+1}^{\varrho}{}_s(s-\alpha)\,\delta g_{2\varrho+1-\alpha-s}\int\overline{H}(xy)_s\,dx$$

$$-\frac{1}{4}\sum_{1}^{\varrho-\alpha}{}_s(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\left\{\sum_{\varrho-s+1}^{\varrho}{}_t\,\mathfrak{g}_{\varrho+s+1,t}\int\overline{H}(xy)_t\,dx\right.$$

$$\left.+\sum_{\varrho-s}^{\varrho}{}_t\,\mathfrak{g}'_{\varrho+s+1,t}\int H'(xy)_t\,dx\right\}$$

$$+\frac{1}{4}(\rho-\alpha+1)\,\delta g_{\varrho-\alpha}J_\varrho$$

$$-\frac{1}{4}\sum_{\alpha}^{2\varrho-\alpha}{}_s(s-\alpha+1)\,g_{2\varrho-\alpha-s}\sum_{1}^{\rho}{}_t\left(\mathfrak{h}_{s,t}J_t + \mathfrak{h}'_{s,t}J'_t\right)$$

Mediae parti haec forma tribui potest:

$$-\frac{1}{4}\sum_{\alpha+1}^{\varrho}{}_t\, J_t \sum_{\varrho-t+1}^{\varrho-\alpha}{}_s\,(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\,\mathfrak{g}_{\varrho+s+1,t}$$

$$-\frac{1}{4}\sum_{\alpha}^{\varrho}{}_t\, J'_t \sum_{\varrho-t}^{\varrho-\alpha}{}_s\,(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\,\mathfrak{g}'_{\varrho+s+1,t}$$

si $-\frac{1}{4}(\rho-\alpha+1)\,\delta g_{\varrho-\alpha}\,\mathfrak{g}'_{\varrho+1,\varrho}\,J'_\varrho$ pro $\frac{1}{4}(\rho-\alpha+1)\,\delta g_{\varrho-\alpha}\,J'_\varrho$ scribitur.

Est igitur:

20) $$\delta J'_\alpha = -\frac{1}{4}\sum_{\alpha+1}^{\varrho}{}_t\,(t-\alpha)\,\delta g_{2\varrho+1-\alpha-t}\,J_t$$

$$-\frac{1}{4}\sum_{\alpha+1}^{\varrho} J_t \sum_{\varrho-t+1}^{\varrho-\alpha}{}_s\,(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\,\mathfrak{g}_{\varrho+s+1,t}$$

$$-\frac{1}{4}\sum_{1}^{\varrho}{}_t\, J_t \sum_{\alpha}^{2\varrho-\alpha}{}_s\,(s-\alpha+1)\,g_{2\varrho-\alpha-s}\,\mathfrak{h}_{s,t}$$

$$-\frac{1}{4}\sum_{\alpha}^{\varrho}{}_t\, J'_t \sum_{\varrho-t}^{\varrho-\alpha}{}_s\,(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\,\mathfrak{g}'_{\varrho+s+1,t}$$

$$-\frac{1}{4}\sum_{1}^{\varrho}{}_t\, J'_t \sum_{\alpha}^{2\varrho-\alpha}{}_s\,(s-\alpha+1)\,g_{2\varrho-\alpha-s}\,\mathfrak{h}'_{s,t}$$

vel etiam

$$\delta J'_\alpha = -\frac{1}{4}\sum_{1}^{\alpha}{}_t\, J_t \sum_{\alpha}^{2\varrho-\alpha}{}_s\,(s-\alpha+1)\,g_{2\varrho-\alpha-s}\,\mathfrak{h}_{s,t}$$

$$-\frac{1}{4}\sum_{\alpha+1}^{\varrho}{}_t\, J_t \left\{(t-\alpha)\,\delta g_{2\varrho+1-\alpha-t} + \sum_{\varrho-t+1}^{\varrho-\alpha}{}_s\,(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\,\mathfrak{g}_{\varrho+s+1,t} + \sum_{\alpha}^{2\varrho-\alpha}{}_s\,(s-\alpha+1)\,g_{2\varrho-\alpha-s}\,\mathfrak{h}_{s,t}\right\}$$

$$-\frac{1}{4}\sum_{1}^{\alpha-1}{}_t\, J'_t \sum_{\alpha}^{2\varrho-\alpha}{}_s\,(s-\alpha+1)\,g_{2\varrho-\alpha-s}\,\mathfrak{h}'_{s,t}$$

$$-\frac{1}{4}\sum_{\alpha}^{\varrho}{}_t\, J'_t \left\{\sum_{\varrho-t}^{\varrho-\alpha}{}_s\,(\rho+s-\alpha+1)\,\delta g_{\varrho-\alpha-s}\,\mathfrak{g}'_{\varrho+s+1,t} + \sum_{\alpha}^{2\varrho-\alpha}{}_s\,(s-\alpha+1)\,g_{2\varrho-\alpha-s}\,\mathfrak{h}'_{s,t}\right\}$$

$(\alpha = 1, 2, 3, \ldots (\rho-1)$

$$\rho = 2.$$

$$\delta J_\beta = \sum_{1}^{2}{}_{\alpha} (\mathfrak{h}_{\beta-1,\alpha}\, J_\alpha + \mathfrak{h}'_{\beta-1,\alpha}\, J'_\alpha)$$

$$(\beta = 1, 2)$$

$$\delta J'_2 = -\sum_{1}^{2}{}_{\alpha} (\mathfrak{h}_{2,\alpha}\, J_\alpha + \mathfrak{h}'_{2,\alpha}\, J'_\alpha)$$

$$\delta J'_1 = -\frac{1}{4}\sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}_{s,1}\, J_1 - \frac{1}{4}\left\{\delta g_2 + \sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}_{s,2}\right\} J_2$$

$$-\frac{1}{4}\sum_{1}^{2}{}_{t}\, J'_t \left\{\sum_{2-t}^{1}{}_{t} (2+s)\, \delta g_{1-s}\, \mathfrak{g}'_{3+s,t} + \sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}'_{s,t}\right\}$$

Quum $\delta g_0 = O$ sit, non reperitur in formula nostra nisi $\mathfrak{g}'_{3,2}$ quod est $= -1$; est igitur:

$$\delta J'_1 = -\frac{1}{4}\sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}_{s,1}\, J_1 - \frac{1}{4}\left\{\delta g_2 + \sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}_{s,2}\right\} J_2 - \frac{1}{4}\sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}'_{s,1}\, J'_1$$

$$-\frac{1}{4}\left\{\sum_{1}^{3}{}_{s}\, s g_{3-s}\, \mathfrak{h}'_{s,2} - 2\,\delta g_1\right\} J'_2$$

Valores quantitatum $\mathfrak{h}_{n,\alpha}$ sunt hi

$$\mathfrak{h}_{n,\alpha} = \sum_{1}^{5}{}_{\mu} \frac{2\, e_\mu{}^n\, H(e_\mu)_\alpha}{R'(e_\mu)}\, \delta e_\mu \qquad (\alpha > n)$$

$$\left.\begin{array}{l}\mathfrak{h}_{n,\alpha} = \sum_{1}^{5}{}_{\mu} \left(\dfrac{2\, e_\mu{}^n\, H(e_\mu)_\alpha}{R'(e_\mu)} + \dfrac{1}{2}\, e_\mu{}^{n-\alpha}\right)\delta e_\mu \qquad (\alpha \leqq n)\end{array}\right\} n = 0, 1, 2$$

$$\mathfrak{h}'_{n,\alpha} = -\sum_{1}^{5}{}_{\mu} \frac{2\, e_\mu{}^{n+\alpha-1}}{R'(e_\mu)}\, \delta e_\mu \qquad (\alpha = 1, 2)$$

et, si $n = 3$

$$\mathfrak{h}_{3,\alpha} = \sum_{1}^{5}{}_{\mu} \left(\frac{1}{2}\, e_\mu{}^{3-\alpha} + \frac{2\, e_\mu{}^3\, H(e_\mu)_\alpha}{R'(e_\mu)}\right)\delta e_\mu \qquad (\alpha = 1, 2)$$

$$\mathfrak{h}'_{3,1} = -\sum_{1}^{5}{}_{\mu} \frac{2 e_\mu{}^3\, \delta e_\mu}{R'(e_\mu)}$$

$$\mathfrak{h}'_{3,2} = \sum_{1}^{5}{}_{\mu} \left\{\frac{1}{2}\, \mathfrak{g}'_{3,2} - \frac{2\, e_\mu{}^4}{R'(e_\mu)}\right\}\delta e_\mu = -\sum_{1}^{5}{}_{\mu} \left(\frac{1}{2} + \frac{2\, e_\mu{}^4}{R'(e_\mu)}\right)\delta e_\mu$$

Est praetera

$$H(x)_1 = -\frac{1}{4}(12\,x^3 + 2\,g_1\,x^2 + g_2\,x)$$

$$H(x)_2 = -\,x^2$$

Aequationes nostrae has finales indunt formas:

$$\delta J_1 = -\sum_1^5{}_\mu\,\delta e_\mu \frac{12\,e_\mu^3 + 2g_1\,e_\mu^2 + g_2\,e_\mu}{2\,R'\,e_\mu}\,J_1 - \sum_1^5{}_\mu\,\delta e_\mu\,\frac{2\,e_\mu^2}{R'e_\mu}\,J_2 - \sum_1^5{}_\mu\,\delta e_\mu \frac{2}{R'e_\mu} J'_1 - \sum_1^5{}_\mu\,\delta e_\mu\,\frac{2\,e_\mu}{R'\,e_\mu} J'_2$$

$$\delta J_1 = -\sum_1^5{}_\mu\,\delta e_\mu \left(\frac{12\,e_\mu^4 + 2g_1\,e_\mu^3 + g_2\,e_\mu^2}{2\,R'\,e_\mu} - \frac{1}{2}\right) J_1 - \sum_1^5{}_\mu\,\delta e_\mu \frac{2e_\mu^3}{R'e_\mu} J_2 - \sum_1^5{}_\mu\,\delta e_\mu \frac{2\,e_\mu}{R'\,e_\mu}\,J'_1 - \sum_1^5{}_\mu\,\delta e_\mu\,\frac{2\,e_\mu^2}{R'\,e_\mu} J'_2$$

$$\delta J'_2 = \sum_1^5{}_\mu\,\delta e_\mu \left(\frac{12e_\mu^5 + 2g_1\,e_\mu^4 + g_2\,e_\mu^3}{2\,R'\,e_\mu} - \frac{1}{2}e_\mu\right) J_1 + \sum_1^5{}_\mu\,\delta e_\mu \left(\frac{2\,e_\mu^4}{R'e_\mu} - \frac{1}{2}\right) J_2 + \sum_1^5{}_\mu\,\delta e_\mu\,\frac{2\,e_\mu^2}{R'e_\mu} J'_1 + \sum_1^5{}_\mu\,\delta e_\mu \frac{2\,e_\mu^3}{R'e_\mu} J'_2$$

$$\delta J'_1 = \frac{1}{4}\sum_1^5{}_\mu\,\delta e_\mu \left(\frac{144\,e_\mu^6 + 48g_1\,e_\mu^5 + (24g_2 + 4g_1^2)e_\mu^4 + 4g_1 g_2\,e_\mu^3 + g_2^2 e_\mu^2}{2\,R'\,e_\mu} - \frac{12e_\mu^2 + 2\,g_1\,e_\mu + g_2}{2}\right) J_1$$

$$+ \frac{1}{4}\left\{\sum_1^5{}_\mu\,\delta e_\mu \left(\frac{24e_\mu^5 + 4g_1\,e_\mu^4 + 2g_2 e_\mu^3}{2R'e_\mu} - \frac{12e_\mu + 2g_1}{2}\right) - \delta g_2\right\} J_2$$

$$+ \frac{1}{4}\sum_1^5{}_\mu\,\delta e_\mu\,\frac{24e_\mu^3 + 4g_1\,e_\mu^2 + 2g_2\,e_\mu}{R'\,e_\mu}\,J'_1$$

$$+ \frac{1}{4}\left\{\sum_1^5{}_\mu\,\delta e_\mu \left(\frac{24e_\mu^4 + 4g_1\,e_\mu^3 + 2\,g_2 e_\mu^2}{R'e_\mu} + 6\right) - 2\,\delta g_1\right\} J'_2$$

$$\rho = 3.$$

$$y^2 = 4\,(x-e_1)\,(x-e_2)\,(x-e_3)\,x-e_4)\,(x-e_5)\,(x-e_6)\,(x-e_7) = R\,(x)$$

$$\delta J_\beta = \sum_{1}^{3}{}_{\alpha} (\mathfrak{h}_{\beta-1,\alpha} J_\alpha + \mathfrak{h}'_{\beta-1,\alpha} J'_\alpha)$$

$$\beta = 1, 2, 3$$

$$\delta J'_3 = -\sum_{1}^{3} (\mathfrak{h}_{3,\alpha} J_\alpha + \mathfrak{h}'_{3,\alpha} J'_\alpha)$$

$$\delta J'_\alpha = -\frac{1}{4}\sum_{1}^{\alpha}{}_t J_t \sum_{\alpha}^{6-\alpha}{}_s (s-\alpha+1)\, g_{6-\alpha-s}\, \mathfrak{h}_{s,t}$$

$$-\frac{1}{4}\sum_{\alpha+1}^{3}{}_t J_t \left\{ (t-\alpha)\,\delta g_{7-\alpha-t} + \sum_{4-t}^{3-\alpha}{}_s (4+s-\alpha)\,\delta g_{3-\alpha-s}\, \mathfrak{g}_{4+s+t,} + \sum_{\alpha}^{6-\alpha}{}_s (s-\alpha+1)\, g_{6-\alpha-s}\, \mathfrak{h}_{s,t} \right\}$$

$$-\frac{1}{4}\sum_{1}^{\alpha-1}{}_t J'_t \sum_{\alpha}^{6-\alpha}{}_s (s-\alpha+1)\, g_{6-\alpha-s}\, \mathfrak{h}'_{s,t}$$

$$-\frac{1}{4}\sum_{\alpha}^{3}{}_t J'_t \left\{ \sum_{\varrho-t}^{3-\alpha}{}_s (\rho+s-\alpha+1)\,\delta g_{3-\alpha-s}\, \mathfrak{g}'_{4+s,t} + \sum_{\alpha}^{6-\alpha}{}_s (s-\alpha+1)\, g_{6-\alpha-s}\, \mathfrak{h}_{s,t} \right\}$$

$$(\alpha = 1, 2)$$

Fst autem $\mathfrak{g}'_{4,3} = 1$

$$\mathfrak{g}'_{5,3} = \frac{1}{6} g_1, \quad \mathfrak{g}'_{5,2} = -\frac{1}{3}, \quad \mathfrak{g}_{5,3} = -\frac{1}{12}\, \mathfrak{g}_2$$

$$\mathfrak{g}'_{6,3} = \frac{9g_2 - 2g_1^2}{60}, \quad \mathfrak{g}'_{6,2} = \frac{1}{15} g_1, \; \mathfrak{g}'_{6,1} = -\frac{1}{5}$$

tum 1) si $n = 0, 1, 2, 3$

$$\mathfrak{h}_{n,\alpha} = \sum_{1}^{7}{}_{\mu} \delta e_\mu \frac{2 e_\mu^n \; H\,(e_\mu)_\alpha}{R'\,(e_\mu)} \qquad (n < \alpha)$$

$$\mathfrak{h}_{n,\alpha} = \sum_{1}^{7}{}_{\mu} \delta e_\mu \left(\frac{2 e_\mu^n \; H\,(e_\mu)_\alpha}{R'\,(e_\mu)} + \frac{1}{2} e_\mu^{n-\alpha} \right) \; (n \geqq \alpha)$$

$$\mathfrak{h}'_{n,\alpha} = -\sum_{1}^{7}{}_{\mu} \delta e_\mu \frac{2 e_\mu^{n+\alpha-1}}{R'\,(e_\mu)}$$

2) si $n = 4, 5, = 3 + m$

$$\mathfrak{h}_{3+m,\alpha} = \sum_{1}^{7}{}_{\mu} \delta e_\mu \left(\frac{2 e_\mu^{3+m} H(e_\mu)_\alpha}{R'(e_\mu)} + \frac{1}{2} e_\mu^{3+m-\alpha} \right)$$

$$\begin{pmatrix} m = 1, & \alpha = 1, 2, 3 \\ m = 2, & \alpha = 1, 2 \end{pmatrix}$$

$$\mathfrak{h}'_{3+m,\alpha} = -\sum_{1}^{7}{}_{\mu} \delta e_\mu \frac{2 e_\mu^{2+m+\alpha}}{R' e_\mu}$$

$$\begin{pmatrix} m = 1, & \alpha = 1, 2 \\ m = 2, & \alpha = 1, \end{pmatrix}$$

$$\mathfrak{h}'_{5,3} = \sum_{1}^{7}{}_{\mu} \delta e_\mu \left(-\frac{1}{12} g_2 + \frac{2 e_\mu^5 H(e_\mu)_3}{R'(e_\mu)} + \frac{1}{2} e_\mu^2 \right)$$

$$\mathfrak{h}'_{3+m,\alpha} = \sum_{1}^{7}{}_{\mu} \delta e_\mu \left\{ \sum_{3-\alpha}^{m-1}{}_{t} \frac{1}{2} \mathfrak{g}'_{4+t,\alpha} e_\mu^{m-t-1} - \frac{2 e_\mu^{2+m+\alpha}}{R'(e_\mu)} \right\}$$

$$m = 1, \quad \alpha = 3$$
$$m = 2, \quad \alpha = 2, 3$$

tum: $H(e_\mu)_1 = -\frac{1}{4} \left(20 e_\mu^5 + 4 g_1 e_\mu^4 + 3 g_2 e_\mu^3 + 2 g_3 e_\mu^2 + g_4 e_\mu \right)$

$$H(e_\mu)_2 = -\frac{1}{4} \left(12 e_\mu^4 + 2 g_1 e_\mu^3 + g_2 e_\mu^2 \right)$$

$$H(e_\mu)_3 = -e_\mu^3$$

Substitutis valoribus quantitatum $\mathfrak{h}_{n,\alpha}$ hae formae finales nobis se praebebunt

$$\delta J_1 = -\sum_\mu \delta e_\mu \frac{20 e_\mu^5 + 4 g_1 e_\mu^4 + 3 g_2 e_\mu^3 + 2 g_3 e_\mu^2 + g_4 e_\mu}{2 R' e_\mu} J_1$$
$$- \sum_\mu \delta e_\mu \frac{12 e_\mu^4 + 2 g_1 e_\mu^3 + g_2 e_\mu^2}{2 R' e_\mu} J_2 - \sum_\mu \delta e_\mu \frac{2 e_\mu^3}{R' e_\mu} J_3$$
$$- \sum_\mu \delta e_\mu \frac{2}{R' e_\mu} J'_1 - \sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu} J'_2 - \sum_\mu \delta e_\mu \frac{2 e_\mu^2}{R' e_\mu} J'_3$$

$$\delta J_2 = -\sum_\mu \delta e_\mu \left(\frac{20 e_\mu^6 + 4 g_1 e_\mu^5 + 3 g_2 e_\mu^4 + 2 g_3 e_\mu^3 + g_4 e_\mu^2}{2 R' e_\mu} - \frac{1}{2} \right) J_1$$
$$- \sum_\mu \delta e_\mu \frac{12 e_\mu^4 + 2 g_1 e_\mu^3 + g_2 e_\mu^2}{2 R' e_\mu} J_2 - \sum_\mu \delta e_\mu \frac{2 e_\mu^4}{R' e_\mu} J_3$$
$$- \sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu} J'_1 - \sum_\mu \delta e_\mu \frac{2 e_\mu^2}{R' e_\mu} J'_2 - \sum_\mu \delta e_\mu \frac{2 e_\mu^3}{R' e_\mu} J'_3$$

$$\delta J_3 = -\sum_\mu \delta e_\mu \left\{ \frac{20 e_\mu^7 + 4 g_1 e_\mu^6 + 3 g_2 e_\mu^5 + 2 g_3 e_\mu^4 + g_4 e_\mu^3}{2 R' e_\mu} - \frac{1}{2} e_\mu \right\} J_1$$

$$- \sum_\mu \delta e_\mu \left(\frac{12 e_\mu^6 + 2 g_1 e_\mu^5 + g_2 e_\mu^4}{2 R' e_\mu} - \frac{1}{2} \right) J_2 - \sum_\mu \delta e_\mu \frac{2 e_\mu^5}{R' e_\mu} J_3$$

$$- \sum_\mu \delta e_\mu \frac{2 e_\mu^2}{R' e_\mu} J'_1 - \sum_\mu \delta e_\mu \frac{2 e_\mu^3}{R' e_\mu} J'_2 - \sum_\mu \delta e_\mu \frac{2 e_\mu^4}{R' e_\mu} J'_3$$

$$\delta J'_3 = \sum_\mu \delta e_\mu \left(\frac{20 e_\mu^8 + 4 g_1 e_\mu^7 + 3 g_2 e_\mu^6 + 2 g_3 e_\mu^5 + g_4 e_\mu^4}{R' e_\mu} - \frac{1}{2} e_\mu^2 \right) J_1$$

$$+ \sum_\mu \delta e_\mu \left(\frac{12 e_\mu^7 + 2 g_1 e_\mu^6 + g_2 e_\mu^5}{2 R' e_\mu} - \frac{1}{2} e_\mu \right) J_2 - \sum_\mu \delta e_\mu \left(\frac{2 e_\mu^6}{R' e_\mu} - \frac{1}{2} \right) J_3$$

$$+ \sum_\mu \delta e_\mu \frac{2 e_\mu^3}{R' e_\mu} J'_1 + \sum_\mu \delta e_\mu \frac{2 e_\mu^4}{R' e_\mu} J'_2 + \sum_\mu \delta e_\mu \frac{2 e_\mu^5}{2 R' e_\mu} J'_3$$

$$\delta J'_2 = \frac{1}{4} \sum_\mu \delta e_\mu \left[\frac{1}{2 R' e_\mu} \left\{ \begin{array}{l} 240 e_\mu^9 + 88 g_1 e_\mu^8 + (56 g_2 + 8 g_1^2) e_\mu^7 + (10 g_1 g_2 + 24 g_3) e_\mu^6 \\ + (4 g_1 g_3 + 3 g_2^2 + 12 g_4) e_\mu^5 + (8 g_1 g_4 + 2 g_2 g_3) e_\mu^4 + g_2 g_4 e_\mu^3 \end{array} \right\} - \frac{12 e_\mu^3 + 2 g_1 e_\mu^2 + g_2 e_\mu}{2} \right] J_1$$

$$+ \frac{1}{4} \sum_\mu \delta e_\mu \left[\frac{1}{2 R' e_\mu} \left\{ 144 e_\mu^8 + 48 g_1 e_\mu^7 + (24 g_2 + 4 g_1^2) e_\mu^6 + 4 g_1 g_2 e_\mu^5 + g_2^2 e_\mu^4 \right\} - \frac{12 e_\mu^2 + 2 g_1 e_\mu + g_2}{2} \right] J_2$$

$$+ \frac{1}{4} \left\{ - \delta g_2 + \sum_\mu \delta e_\mu \left[\frac{24 e_\mu^7 + 4 g_1 e_\mu^6 + 2 g_2 e_\mu^5}{R' e_\mu} - \frac{12 e_\mu + 2 g_1}{2} \right] J_3 \right.$$

$$+ \frac{1}{4} \sum_\mu \delta e_\mu \frac{24 e_\mu^4 + 4 g_1 e_\mu^3 + 2 g_2 e_\mu^2}{R' e_\mu} J'_1$$

$$+ \frac{1}{4} \sum_\mu \delta e_\mu \frac{24 e_\mu^5 + 4 g_1 e_\mu^4 + 2 g_2 e_\mu^3}{R' e_\mu} J'_2$$

$$+ \frac{1}{4} \left\{ 2 \delta g_1 + \sum_\mu \delta e_\mu \left[\frac{24 e_\mu^6 + 4 g_1 e_\mu^5 + g_2 e_\mu^4}{R' e_\mu} + 6 \right] \right\} J'_3$$

$$\delta J'_1 = \frac{1}{4} \sum_\mu \delta e_\mu \left[\frac{1}{R' e_\mu} \left\{ 400 e_\mu^{10} + 160 g_1 e_\mu^9 + (120 g_2 + 16 g_1^2) e_\mu^8 + (80 g_3 + 24 g_1 g_2) e_\mu^7 + (40 g_4 + 16 g_1 g_3 + 9 g_2^2) e_\mu^6 + (8 g_1 g_4 + 12 g_2 g_3) e_\mu^5 + (3 g_2 g_4 + 4 g_3^2) e_\mu^4 + 4 g_4 g_3 e_\mu^3 + g_4 e_\mu^2 \right\} - \frac{1}{12} (20 e_\mu^4 + 4 g_1 e_\mu^3 + 3 g_2 e_\mu^2 + 2 g_3 e_\mu + g_4) \right] J_1$$

$$+ \frac{1}{4} \left\{ - \delta g_4 + \sum_\mu \delta e_\mu \left[\frac{1}{R' e_\mu} \left\{ 240 e_\mu^9 + 88 g_1 e_\mu^8 + (56 g_2 + 8 g_1^2) e_\mu^7 + (10 g_1 g_2 + 24 g_3) e_\mu^6 + (3 g_2^2 + 4 g_1 g_3 + 12 g_4) e_\mu^5 + (2 g_2 g_3 + 2 g_1 g_4) e_\mu^4 + g_2 g_4 e_\mu^3 \right\} - \frac{20 e_\mu^3 + 4 g_1 e_\mu^2 + 3 g_2 e_\mu + 2 g_3}{2} \right] \right\} J_2$$

$$+ \frac{1}{4} \left\{ - 2 \delta g_3 + \frac{1}{3} g_2 \delta g_1 + \sum_\mu \delta e_\mu \left[\frac{40 e_\mu^8 + 8 g_1 e_\mu^7 + 6 g_2 e_\mu^6 + 4 g_3 e_\mu^5 + 2 g_4 e_\mu^4}{R' e_\mu} - \frac{20 e_\mu^2 + 4 g_1 e_\mu + 3 g_2}{2} + \frac{5}{3} g_2 \right] \right\} J_3$$

$$+ \frac{1}{4} \sum_\mu \delta e_\mu \frac{40 e_\mu^5 + 8 g_1 e_\mu^4 + 6 g_2 e_\mu^3 + 4 g_3 e_\mu^2 + 2 g_4 e_\mu}{R' e_\mu} J'_1$$

$$+ \frac{1}{4} \left\{ \frac{4}{3} \delta g_1 + \sum_\mu \delta e_\mu \left[\frac{40 e_\mu^6 + 8 g_1 e_\mu^5 + 6 g_2 e_\mu^4 + 4 g_3 e_\mu^3 + 2 g_4 e_\mu^2}{R' e_\mu} + \frac{10}{3} \right] \right\} J'_2$$

$$+ \frac{1}{4} \left\{ 3 \delta g_2 - \frac{2}{3} g_1 \delta g_1 + \sum_\mu \delta e_\mu \left[\frac{40 e_\mu^7 + 8 g_1 e_\mu^6 + 6 g_2 e_\mu^5 + 4 g_3 e_\mu^4 + 2 g_4 e_\mu^3}{R' e_\mu} + \frac{20 e_\mu + 2 g_1}{2} + \frac{10}{3} \right] \right\} J'_3$$

Formulae hac altera methodo inventae symmetras tantum continent functiones radicum e_α aequationis $Rx = O$, et propter hoc rationaliter per coëfficientes g_α exprimi possunt. Si hoc calculare velis formulas priore methodo inventas denuo invenias. Quod facile fieri potest, si $\rho = 1$ ponatur.

Prima methodus suppeditat nobis has aequationes differentiales:

$$\delta J = \sum_{o}^{2}{}_r \, \delta g_{3-r} \frac{-h_r}{2L} J + \sum_{o}^{2}{}_r \, \delta g_{3-r} \frac{-h'_r}{2L} J'$$

$$\delta J = \sum_{o}^{2}{}_r \, \delta g_{3-r} \frac{h_{r+1}}{2L} J + \sum_{o}^{2}{}_r \, \delta g_{3-r} \frac{h'_{r+1}}{2L} J'$$

Est autem $H(xy) = \frac{1}{y}$, $H'(xy) = -\frac{x}{y}$

$$h_r = -4 A_{r+1}, \quad h'_r = -4 A_r$$

$$a_{r,s} = (r+1) g_{2-r} g_{2-s} - \sum_{1}^{r+1}{}_n (s-r+2n) g_{2-r+n} \, g_{2-s-n} \quad (s \geqq r)$$

$$= (s+1) g_{2-r} g_{2-s} \quad \sum_{1}^{s+1}{}_n (r-s+2n) g_{2-s+n} g_{2-r-n} \quad s < r$$

$$L = \sum \pm a_{00} \, a_{11} \, a_{22} = \begin{vmatrix} g_2^2 - 2g_3 g_1 & g_1 g_2 - 12 g_3 & 4 g_2 \\ g_1 g_2 - 12 g_3 & 2 g_1^2 - 8 g_2 & 8 g_1 \\ 4 g_2 & 8 g_1 & 48 \end{vmatrix} \quad A_{r+s} = \frac{dL}{da_{r,s}}$$

Ponatur $g_1 = o$ et $-g_2$, $-g_3$ pro $+g_2$, $+g_3$ erit:

$$L = 256 \, (g_2^3 - 27 g_3^2)$$

$$h_o = 4.576 g_3 \qquad h'_o = -4.384 g_2$$

$$h_1 = -128 g_2^2 \qquad h'_1 = -4.576 g_3$$

$$h_2 - 192 g_2 g_3 \qquad h'_2 = -4.32 g_2^2$$

$$\delta J = \frac{18 g_3 \delta g_3 - g_2^2 \delta g_2}{8(g_2^3 - 27 g_3^2)} J + \frac{-12 g_2 \delta g_3 + 18 g_3 \delta g_2}{4(g_2^3 - 27 g_3^2)} J'$$

$$\delta J' = \frac{2 g_2^2 \delta g_3 - 3 g_2 g_3 \delta g_2}{8(g_2^3 - 27 g_3^2)} J + \frac{-18 g_3 \delta g_3 + g_2^2 \delta g_2}{4(g_2^3 - 27 g_3^2)} J'$$

Secundum alteram methodum erit:

$$\delta \mathrm{J} = \mathfrak{h}_o \mathrm{J} + \mathfrak{h}'_o \mathrm{J}', \quad \mathfrak{h}_o = \sum_\mu \delta e_\mu \frac{2H(e_\mu)}{R' e_\mu} = -\sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu}, \qquad \mathfrak{h}'_o = -\sum_\mu \delta e_\mu \frac{2}{R'(e_\mu)}$$

$$\delta \mathrm{J}' = -\mathfrak{h}_1 \mathrm{J} - \mathfrak{h}'_1 \mathrm{J}'_1, \mathfrak{h}_1 = -\sum_\mu \delta e_\mu \frac{2 e_\mu^2}{R' e_\mu} \qquad \mathfrak{h}'_1 = -\sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu}$$

$$\delta \mathrm{J} = -\sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu} \mathrm{J} - \sum_\mu \delta e_\mu \frac{2}{R' e_\mu} \mathrm{J}'$$

$$\delta \mathrm{J}' = \sum_\mu \delta e_\mu \frac{2 e_\mu^2}{R' e_\mu} \mathrm{J} + \sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu} \mathrm{J}'$$

Habemus autem primum:

$$e_\alpha + e_\beta + e_\gamma = g_1 = 0 \qquad \delta e_\alpha + \delta e_\beta + \delta e_\gamma = 0$$

$$e_\alpha e_\beta + e_\beta e_\gamma + e_\gamma e_\alpha = -\frac{g_2}{4}, \quad e_\alpha \delta e_\alpha + e_\beta \delta e_\beta + e_\gamma \delta e_\gamma = \frac{\delta g_2}{4}, \quad \begin{vmatrix} 1 & 1 & 1 \\ e_\alpha & e_\beta & e_\gamma \\ e_\beta e_\gamma, & e_\gamma e_\alpha, & e_\alpha e_\beta \end{vmatrix} = \begin{cases} (e_\alpha - e_\beta)\cdot \\ (e_\beta - e_\gamma)\cdot \\ (e_\gamma - e_\alpha) \end{cases} = P$$

$$e_\alpha e_\beta e_\gamma = \frac{g_3}{4}, \qquad e_\alpha e_\beta \delta e_\gamma + e_\beta e_\gamma \delta e_\alpha + e_\gamma e_\alpha \delta e_\beta = \frac{\delta g_3}{4},$$

$$\delta e_\alpha = \frac{e_\alpha \delta g_2 + \delta g_3}{P} \, \frac{e_\gamma - e_\beta}{4} = \frac{e_\alpha \delta g_2 + \delta g_3}{R'(e_\alpha)}$$

$$\sum_\alpha \frac{e_\alpha^n \delta e_\alpha}{R' e_\alpha} = \sum_\alpha \frac{e_\alpha^{n+1} \delta g_2 + e_\alpha^n \delta g_3}{R'^2(e_\alpha)} = \delta g_2 \sum_\alpha \frac{e_\alpha^{n+1}}{R'^2(e_\alpha)} + \delta g_3 \sum_\alpha \frac{e_\alpha^n}{R'^2(e_\alpha)}$$

$$\sum_\alpha \frac{e_\alpha^n}{R'^2(e_\alpha)} = \sum_\alpha \frac{e_\alpha^n R'^2(e_\beta) R'^2(e_\gamma)}{(\Pi_\nu R'(e_\nu)^2)} = \sum_\alpha \frac{e_\alpha^n 4^4 P^2 (e_\beta - e_\gamma)^2}{\Pi_\nu R'(e_\nu). \ \Pi_\nu R' e_\nu}$$

est enim $R'(e_\beta) R'(e_\gamma) = 4.4\,(e_\beta - e_\alpha)(e_\beta - e_\gamma)(e_\gamma - e_\alpha)(e_\gamma - e_\beta) = 4.4\,P(e_\beta - e_\gamma)$
et $\Pi_\nu R'(e_\nu) = -4^3 (e_\alpha - e_\beta)^2 (e_\beta - e_\gamma)^2 (e_\gamma - e_\alpha)^2 = -4^3 P^2$

$$\sum_\alpha \frac{e_\alpha^n}{(R' e_\alpha)^2} = \sum \frac{e_\alpha^n 4^4 P^2 (e_\beta - e_\gamma)^2}{4^6 P^4} = \sum_\alpha \frac{e_\alpha^n (e_\beta - e_\gamma)^2}{4.4\,P^2}$$

Est $R(x) = 4(x - e_\alpha)(x - e_\beta)(x - e_\gamma) = 4x^3 - g_2 x - g_3$

Sit $R_1(x) = \dfrac{R x}{x - e_\alpha} = 4(x - e_\beta)(x - e_\gamma) = 4x^2 + 4e_\alpha x + 4e_\alpha^2 - g_2$

$$\begin{aligned} (e_\beta - e_\gamma)^2 &= (e_\beta + e_\gamma)^2 - 4 e_\beta e_\gamma \\ &= e_\alpha^2 - 4e_\alpha^2 + g_2 = g_2 - 3e_\alpha^2 \end{aligned}$$

$$\sum_\alpha \frac{e_\alpha^n}{(R' e_\alpha)^2} = \frac{1}{16 P^2} \sum_\alpha e_\alpha^n (e_\beta - e_\gamma)^2 = \frac{1}{16 P^2} \sum_\alpha \left(g_2 e_\alpha^n - 3 e_\alpha^{n+2} \right)$$

$$= \frac{1}{16 P^2} \left\{ g_2 \sum_\alpha e_\alpha^n - 3 \sum_\alpha e_\alpha^{n+2} \right\}$$

Habemus generaliter, si $X = (x - a_1)(x - a_2) \ldots (x - a_n)$ et $X' = \dfrac{dX}{dx}$

$$\frac{x X'}{X} = \frac{x}{x - a_1} + \frac{x}{x - a_2} + \ldots + \frac{x}{x - a_n} = n + \sum_\alpha a_\alpha x^{-1} + \sum_\alpha a_\alpha^2 x^{-2} + \ldots + \sum_\alpha a_\alpha^n x^{-n} + \ldots$$

$$\frac{x R'(x)}{R x} = \frac{12 x^3 - g_2 x}{4x^3 - g_2 x - g_3} = 3 + \frac{1}{2} g_2 x^{-2} + \frac{3}{4} g_3 x^{-3} + \frac{1}{8} g_2^2 x^{-4} + \frac{5}{16} g_2 g_3 x^{-5} + \ldots$$

$$\sum_\alpha e_\alpha^0 = 3, \qquad \sum_\alpha e_\alpha = 0, \quad \sum_\alpha e_\alpha^2 = \frac{1}{2} g_2,$$

$$\sum_\alpha e_\alpha^3 = \frac{3}{4} g_3, \quad \sum_\alpha e_\alpha^4 = \frac{1}{8} g_2^2, \quad \sum_\alpha e_\alpha^5 = \frac{5}{16} g_2 g_3, \ldots$$

$$\sum_\mu \frac{e_\mu^n \,\delta e_\mu}{R'(e_\mu)} = \frac{\delta g_2}{16 P^2} \left\{ g_2 \sum_\mu e_\mu^{n+1} - 3 \sum_\mu e_\mu^{n+3} \right\}$$

$$+ \frac{\delta g_3}{16 P^2} \left\{ g_2 \sum_\mu e_\mu^{n} - 3 \sum_\mu e_\mu^{n+2} \right\} \qquad n = 0, 1, 2$$

$$\sum_\mu \frac{\delta e_\mu}{R' e_\mu} = \frac{1}{16 P^2} \left\{ \delta g_2 \left(g_2 \sum_\mu e_\mu - 3 \sum e_\mu^3 \right) + \delta g_3 \left(g_2 \sum e_\mu^0 - 3 \sum_\mu e_\mu^2 \right\} \right.$$

$$= \frac{1}{16 P^2} \left\{ - \frac{9}{4} g_3 \delta g_2 + \delta g_3 \left(3 g_2 - \frac{3}{2} g_2 \right) \right\} = \frac{-\frac{9}{4} g_3 \delta g_2 + g_{\frac{6}{4}2} \delta g_3}{16 P^2}$$

$$16 P^2 \sum_\mu \frac{e_\mu \,\delta e_\mu}{R' e_\mu} = \delta g_2 \left\{ g_2 \sum_\mu e_\mu^2 - 3 \sum_\mu e_\mu^4 \right\} + \delta g_3 \left(g_2 \sum_\mu e_\mu - 3 \sum_\mu e_\mu^3 \right)$$

$$= \delta g_2 \left(\frac{1}{2} g_2^2 - \frac{3}{8} g_2^2 \right) - \frac{9}{4} g_3 \delta g_3 = \frac{g_2^2 \delta g_2 - 18 g_3 \delta g_3}{8}$$

$$16 P^2 \sum_\mu \frac{e_\mu^2 \,\delta e_\mu}{R' e_\mu} = \delta g_2 \left\{ g_2 \sum_\mu e_\mu^3 - 3 \sum_\mu e_\mu^5 \right\} + \delta g_3 \left\{ g_2 \sum_\mu e_\mu^2 - 3 \sum_\mu e_\mu^4 \right\}$$

$$= \delta g_2 \left\{ \frac{3}{4} g_2 g_3 - \frac{15}{16} g_2 g_3 \right\} + \delta g_3 \left\{ \frac{1}{2} g_2^2 - \frac{3}{8} g_2^2 \right\}$$

$$= \frac{-3 g_2 g_3 \delta g_2 + 2 g_2^2 \delta g_3}{16}$$

$$P^2 = (e_\alpha - e_\beta)^2 (e_\beta - e_\gamma)^2 (e_\gamma - e_\alpha)^2 = (g_2 - 3 e_\gamma^2)(g_2 - 3 e_\beta^2)(g_2 - 3 e_\alpha^2)$$

$$= g_2^3 - 3 g_2^2 (e_\alpha^2 + e_\beta^2 + e_\gamma^2) + 9 g_2 (e_\alpha^2 e_\beta^2 + e_\beta^2 e_\gamma^2 + e_\gamma^2 e_\alpha^2) - 27 e_\alpha^2 e_\beta^2 e_\gamma^2$$

$$= g_2^3 - \frac{3}{2} g_2^3 + \frac{9 g_2^3}{16} - 27 \frac{g_3^2}{16} = \frac{g_2^3 - 27 g_3^2}{16} \qquad \left(\begin{aligned} \Sigma e_\alpha^2 e_\beta^2 &= (\Sigma e_\alpha e_\beta)^2 - 2 (\Sigma e_\alpha) e_\alpha e_\beta e_\gamma \\ &= (\Sigma e_\alpha e_\beta)^2 = \tfrac{1}{16} g_2^2 \end{aligned} \right)$$

et $16 P^2 = g_2^3 - 27 g_3^2$

Ergo:

$$- \sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu} = \frac{- g_2^2 \delta g_2 + 18 g_3 \delta g_3}{4 (g_2^3 - 27 g_3^2)}, \quad - \sum_\mu \delta e_\mu \frac{2}{R' e_\mu} = \frac{18 g_3 \delta g_2 - 12 g_2 \delta g_3}{4 (g_2^3 - 27 g_3^2)}$$

$$\sum_\mu \delta e_\mu \frac{2 e_\mu}{R' e_\mu} = \frac{-3 g_2 g_3 \delta g_2 + 2 g_2^2 \delta g_3}{8 (g_2^3 - 27 g_3^2)}$$

q. e. d.

II.

Inventis hoc modo aequationibus differentialibus ad periodos integralium hyperellipticorum definiendas, exponamus nunc breviter, quomodo aequationes similes formandae sint, si de integralibus abelianis generalibus agitur, pro quibus $\rho = 3$ sit.

Proficiscamur ab illa aequatione, quae normalis vocatur:

$$y^3 - \beta_1 y^2 + \alpha_1 \alpha_3 y - \alpha_1^2 \beta_3 = O$$

ubi α_1, α_3, β_1, β_3 functiones sunt solius x, quarum gradus sunt resp. 1, 2, 1, 3. Aequatio haec hac fruitur proprietate: pro magnis valoribus x unam solam esse repraesentationem pro y. Scribantur η, ξ pro y, x, erit aequatio nostra explicite scripta haec:

$$\eta^3 - (g\xi + h)\eta^2 + (k\xi + l)(m\xi^2 + n\xi + p)\eta - (k\xi + l)^2 (q\xi^3 + r\xi^2 + s\xi + t) = O$$

Transformatione rationali aequatio haec facile ad aequationem quartae dimensionis revocatur; ordinem ρ autem transformatione hac non mutari scimus.

Ponamus $\eta = (\alpha y + \beta) x$, $\xi = \gamma x + \delta$ vel $y = \dfrac{\gamma\eta - \beta(\xi - \delta)}{\alpha(\xi - \delta)}$, $x = \dfrac{\xi - \delta}{\gamma}$

habebimus:

$$\left.\begin{aligned}&(\alpha^3 y^3 + 3\alpha^2\beta y^2 + 3\alpha\beta^2 y + \beta^3)x^3 - (g\gamma x + g\delta + h)(\alpha^2 y^2 + 2\alpha\beta y + \beta^2)x^2 \\ &+ (k\gamma x + k\delta + l)(m\gamma^2 x^2 + (2m\gamma\delta + n\gamma)x + m\delta^2 + n\delta + p)(\alpha y + \beta)x \\ &- (k\gamma x + k\delta + l)^2 \left\{ q(\gamma^3 x^3 + 3\gamma^2\delta x^2 + 3\gamma\delta^2 x + \delta^3) + r(\gamma^2 x^2 + 2\gamma\delta x + \delta^2) \right. \\ &\qquad\qquad \left. + s\gamma x + s\delta + t \right\}\end{aligned}\right| = O$$

vel etiam

$$\left.\begin{aligned}&y^3\alpha^3 x^3 + y^2\big((3\beta - g\gamma)x - (g\delta + h)\big)\alpha^2 x^2 \\ &+ y\alpha\big((3\beta^2 - 2g\beta\gamma)x - 2\beta(g\delta + h) + k\gamma(m\gamma^2 x^2 + (2m\gamma\delta + n\gamma)x + m\delta^2 + n\delta + p)\big) \\ &+ \beta^3 x^3 - \beta^2 g\gamma x^3 - \beta^2(g\delta + h)x^2 + \beta x\big(k\gamma x + (k\delta + l)\big)\big(m\gamma^2 x^2 + (2m\gamma\delta + n\gamma)x \\ &\qquad\qquad + m\delta^2 + n\delta + p)\big) \\ &- (k\gamma x + k\delta + l)^2 \left\{ q(\gamma^3 x^3 + 3\gamma^2\delta x^2 + 3\gamma\delta^2 x + \delta^3) + r(\gamma^2 x^2 + 2\gamma\delta x + \delta^2) \right. \\ &\qquad\qquad \left. + s\gamma x + s\delta + t \right\}\end{aligned}\right| = O$$

Ponamus primum $k\delta + l = O$; quum factor communis x^2 ommitti possit, erit:

$$\left.\begin{array}{l} y^3\alpha^3 x + y^2\left((3\beta - g\gamma)x - g\delta - h\right)\alpha^2 + \\ \quad y\alpha\left((3\beta^2 - 2g\beta\gamma)x - 2\beta(g\delta + h) + k\gamma(m\gamma^2x^2 + (2m\gamma + n\gamma)x + m\delta^2 + n\delta + p)\right. \\ + \beta^2(\beta - g\gamma)x + \beta^2(g\delta + h) + \beta\gamma k(m\gamma^2x^2 + (2m\gamma\delta + n\gamma)x + m\delta^2 + n\delta + p) \\ - k^2\gamma^2\{q\gamma^3x^3 + (3q\gamma^2\delta + r\gamma^2)x^2 + (3q\gamma\delta^2 + 2r\gamma\delta + s\gamma)x + q\delta^3 + r\delta^2 + s\delta + t\} \end{array}\right\} = O$$

Aequentur nunc inter se coëfficientes potestatum: y^3x, yx^2, x^3:

$$\alpha^3 = \alpha k m\gamma^3 = -k^2 q\gamma^5,$$

praeterea ponatur

$$\beta k m\gamma^3 - k^2\gamma^2(3q\gamma^2\delta + r\gamma^2) = O$$

Ultima aequatio etiam tali modo scribi potest:

$$\beta m - \gamma k(3q\delta + r) = O$$

$$\beta = \frac{\gamma k}{m}(3q\delta + r)$$

tum est

$$\alpha^2 = k m\gamma^3$$

$$\frac{\alpha m = -kq\gamma^2}{O = k m^3\gamma^3 - k^2 q^2\gamma^4}$$

$$\gamma = \frac{m^3}{kq^2}, \qquad \alpha = -\frac{1}{m}\,kq\,\frac{m^6}{k^2q^4} = -\frac{m^5}{kq^3}, \qquad \delta = -\frac{l}{k}$$

$$\beta = \frac{m^3 k}{kq^2 m}(3q\delta + r) = \frac{m^2}{q^2}\left(3q\left(-\frac{l}{k}\right) + r\right)$$

$$\beta = \frac{m^2(rk - 3lq)}{q^2k}$$

Est igitur aequatio nostra

$$\left.\begin{array}{l} y^3x\alpha^3 + y^2\left((3\beta - g\gamma)x - g\delta - h\right)\alpha^2 \\ \qquad + y\left(\alpha^3x^2 + \alpha x(3\beta^2 - 2g\beta\gamma + 2m\gamma\delta + n\gamma) - 2\beta(g\delta + h) + m\delta^2 + n\delta + p\right) \\ - k^2q\gamma^5x^3 + \left(\beta^2(\beta - \gamma g) - k^2\gamma^2(3q\gamma\delta^2 + 2r\gamma\delta + s\gamma) + \beta\gamma k(2m\gamma\delta + n\gamma)\right)x \\ \qquad + \left(\beta^2(g\delta + h) + (m\delta^2 + n\delta + p)\beta g k - k^2\gamma^2(q\delta^3 + r\delta^2 + s\delta + t)\right) \end{array}\right\} = O$$

Jam hic substituendi sunt valores $\alpha, \beta, \gamma, \delta$, tum autem dividenda est aequatio per $\alpha^3 = -\frac{m^{15}}{k^3q^9}$. Hoc modo nova orietur aequatio hujus formae:

$$y^3x + y^2(g_{1,2}x + g_{0,2}) + y(x_2 + g_{1,1}x + g_{0,1}) + x^3 + g_{1,0}x + g_{0,0} = O$$

vel, si brevitatis causa ponitur

$$g_{1,2} = a \qquad g_{0,2} = d$$
$$g_{1,1} = b \qquad g_{0,1} = g$$
$$g_{1,0} = c \qquad g_{0,0} = k$$

1) $$y^3x + y^2(ax + d) + y(x^2 + bx + g) + x^3 + cx + k = O$$

Pro magnis valoribus x et y has habemus repraesentationes:

2) $\bar{x}_t = t^{-3}$

$$\bar{y}_t = -t^{-2} + \frac{1}{3}t^{-1} - \frac{a}{3} + \frac{9a-1}{81}t + \frac{-27a^2+9a+81b-1}{243}t^2 - \frac{d}{3}t^3$$
$$+ \frac{-162a^3+162a^2-45a+729ab-243b-2187c+729d+4}{6561}t^4$$
$$+ \frac{-405a^3+270a^2+1458ab-4374ad-63a-324b-2187c+729d+6561g+5}{19683}t^5$$
$$+ t^7\mathfrak{P}(t)$$

vel $= \bar{y}_t = -t^{-2} + \alpha t^{-1} + \beta + \gamma t + \delta t^2 + \varepsilon t^3 + \zeta t^4 + \vartheta t^5 + \lambda t^6 + \ldots$ (sed $\lambda = O$)

Tum autem ponamus simili modo:

$$\bar{y}_t^2 = t^{-4} + \alpha' t^{-3} + \beta' t^{-2} + \gamma' t^{-1} + \delta' + \varepsilon' t + \zeta' t^2 + \vartheta' t^3 + \lambda' t^4 + \ldots$$
$$= t^{-4} - \frac{2}{3}t^{-3} + \frac{6a+1}{9}t^{-2} + \frac{2-36a}{81}t^{-1} + \frac{a^2-2b}{3}$$
$$+ \frac{-108a_2+24a+162b+486d-2}{729}t$$
$$+ \frac{810a^3-405a^2+90a-2916ab-4374ad+486b+4374c-1458d-7}{6561}t^2$$
$$+ \frac{2ad-2g}{3}t^3$$
$$+ \left\{\frac{1701a^4-2268a^3-8748a^2b+945a^2+5832ab+6561b^2+13122ac-17496ad}{59049} \atop \frac{-168a-810b-4374c+1944d+13122g+11}{59049}\right\} t^4$$
$$+ \ldots\ldots$$

Fit y etiam infinitum pro $x = O$. Hoc loco hanc habemus repraesentationem:

3) $$y = -\frac{d}{x} + \frac{g-ad}{d} + \frac{g^2-adg+bd^2-dk}{d^3}x + \ldots.$$

Si formamus functiones, quae non fiunt infinitae, nisi quum $x = \infty$, $y = \infty$ est, statim intelligemus, non posse formari functiones ordinis 1, 2, 4. Ipsum x simplicissima est functio omnium quaesitarum functionum est autem ordinis 3^{tii} Subsequitur illam functio ordinis 5^{ti} quae est $\xi_5 = xy$. Tum erit ordinis 7^{mi} functio haec:

$$\xi_7 = y^2x + dy$$

Est enim pro parvis valoribus x

$$y^2 = \frac{d^2}{x^2} + \frac{2ad-2g}{x} + \ldots.$$

$$xy^2 = \frac{d^2}{x} + (2ad-2g) + x\mathfrak{P}(x)$$

$$dy = -\frac{d^2}{x} + (g-ad) + x\mathfrak{P}_1(x)$$

$xy^2 + dy = (ad - g) + x\,\mathfrak{P}_2\,(x)$ id est: nostra functio non fit infinita, si $x = O$. Ponatur $x = \xi_3$.

Omnis alia functio hujusmodi ex functionibus ξ_3, ξ_5, ξ_7 componitur per additionem et multiplicationem, et hac forma continetur:

$$P\,(\xi_3)\,\xi_7 + Q\,(\xi_3)\,\xi_5 + R\,(\xi_3)$$

ubi P, Q, R integrae sunt functiones solius ξ_3.

His praemissis jam ad functiones illas $H\,(xy)_\alpha$ et $H'\,(xy)_\alpha$ construendas adeamus, quae integralibus primae ac secundae speciei formandis inserviunt. Et hic etiam, ut in theoria integralium hyperellipticorum, formanda est primum functio $H\,(xy\,x'y')$, quae, si prius par argumentorum intuearis, fit infinita primi ordinis si $x = x'$, $y = y'$ et praeterea si $x = \infty$ et $y = \infty$. Pro his valoribus $x = \infty$, $y = \infty$ evoluta non continet functio $H\,(xy\,x'\,y')$ alias potestates negativas nisi 1, 2, 4, quarum coëfficientes quaesitas functiones $H\,(x\,y)_\alpha$ nobis suppeditant. Functiones H'_α autem facile ex funtionibus $\overset{k\alpha}{H}$ formantur, quae sunt coëfficientes positivarum potestatum 1, 2, 4 nostrae evolutionis.

Sit $G\,(xy) = O$ aequatio nostra inter x et y; ponatur

$$G\,(x\,y\,y') = \frac{G\,(xy') - G\,(xy)}{y' - y}$$

ubi x, y valores respondentes, y' autem aliquis alius valor est, qui una cum x' aequationi $G\,(x'\,y') = O$ satisfacit. In formanda functione $H\,(x\,y\,x'\,y')$ necesse est, ut supponamus, non esse y' radicem multiplicem aequationis nostrae; est igitur $G\,(x'y')_2 = \dfrac{\partial\,G\,(x'y')}{\partial\,y'} \gtrless O$.
Itaque est

$$G\,(xyy') = \frac{x\,(y'^3 - y^3) + (ax + d)\,(y'^2 - y^2) + (x^2 + bx + g)\,(y' - y)}{y' - y}$$

$$= x\,(y'^2 + y\,y' + y^2) + (a\,x + d)\,(y' + y) + x^2 + bx + g$$

$$\frac{G\,(x\,y\,y')}{(x' - x)\;G\,(x'y')_2} = \frac{x\,(y'^2 + yy' + y^2) + (a\,x + d)\,(y' + y) + x^2 + bx + g}{(x' - x)\;G\,(x'\,y')_2}$$

Ex hac functione oritur functio $H\,(xy\,x'\,y')$, si ipsa ita transformatur, ut pro $x = \infty$, $y = \infty$ evoluta negativas argumenti potestates has tantum contineat: 1, 2, 4 simul autem membro constanti careat.

Habemus autem

$$\frac{1}{x' - x} = -\left(\frac{1}{x} + \frac{x'}{x^2} + \frac{x'^2}{x^3} + \ldots\right)$$

et

$$\frac{1}{x' - x_t} = -\,(t^3 + x'\,t^6 + x'^2\,t^9 + \ldots .)$$

et praeterea, si secundum ordinem singulorum membrorum scribimus:

$$\begin{aligned} G(\bar{x}_t, \bar{y}_t, y') &= \bar{x}_t\bar{y}_t^2 + \bar{x}_t^2 + \bar{x}_t\bar{y}_t(y'+a) + \bar{x}_t(y'^2+ay'+b) + d\,\bar{y}_t + dy' + g \\ &= t^{-7} + \alpha' t^{-6} + \beta' t^{-5} + \gamma' t^{-4} + \delta' t^{-3} + \varepsilon' t^{-2} + \zeta' t^{-1} + \vartheta' + \lambda' t + \dots \\ &\quad + t^{-6} \\ &\quad + (y'+a)(-t^{-5} + \alpha t^{-4} + \beta t^{-3} + \gamma t^{-2} + \delta t^{-1} + \varepsilon + \zeta t + \dots) \\ &\qquad + (y'^2 + ay' + b)\, t^{-3} \\ &\qquad\qquad + d(-t^{-2} + \alpha t^{-1} + \beta + \gamma t + \dots) \\ &\qquad\qquad + (dy' + g) \\ &= t^{-7} + (\alpha'+1)t^{-6} + (-y'-a+\beta')t^{-5} + (\gamma' + \alpha(y'+a))t^{-4} \\ &\qquad + (\delta' + \beta(y'+a) + y'^2 + ay' + b)\, t^{-3} \\ &\quad + (\varepsilon' + \gamma(y'+a) - d)t^{-2} + (\zeta' + \delta(y'+a) + \alpha d)t^{-1} \\ &\quad + (\vartheta' + \varepsilon(y'+a) + \beta d + dy' + g) \\ &\quad + (\lambda' + \zeta(y'+a) + d\gamma)\, t + \dots \end{aligned}$$

Tum erit:

$$\begin{aligned} \frac{G(x_t\, y_t\, y')}{x' - x_t} = -\Big\{ & t^{-4} + (\alpha'+1)t^{-3} + (\beta'-y'-a)t^{-2} + \left.\begin{matrix}(\gamma' + \alpha(y'+a)) \\ x'\end{matrix}\right| t^{-1} \\ & + \begin{matrix}(\delta' + \beta(y'+a) + y'^2 + ay' + b) \\ (\alpha'+1)x'\end{matrix} \\ & + \left.\begin{matrix}(\varepsilon' + \gamma(y'+a) - d) \\ x'(-y'-a+\beta')\end{matrix}\right| t + \left.\begin{matrix}(\zeta' + \delta(y'+a) + \alpha d) \\ (\gamma' + \alpha(y'+a))x' \\ x'^2\end{matrix}\right| t^2 + \left.\begin{matrix}(\vartheta' + \varepsilon(y'+a) + \beta d + dy' + g) \\ (\delta' + \beta(y'+a) + y'^2 + ay' + b)x' \\ (\alpha'+1)x'^2\end{matrix}\right| t^3 \\ & + \left.\begin{matrix}(\lambda' + \zeta(y'+a) + d\gamma) \\ (\varepsilon' + \gamma(y'+a) - d)x' \\ (-y'-a+\beta')x'^2\end{matrix}\right| t^4 + \dots \Big\} \end{aligned}$$

Tum fiet:

$$\begin{aligned} \frac{G(\bar{x}_t\, \bar{y}_t\, y')}{(x' - x_t)\, G(x'y')_2} = -\frac{1}{G(x'y')_2} \Big\{ & t^{-4} + (\alpha'+1)t^{-3} + (\beta' - y' - a)t^{-2} + (\gamma' + \alpha(y'+a) + x')t^{-1} \\ & + (\delta' + \beta(y'+a) + y'^2 + ay' + b + (\alpha'+1)x') \\ & + (\varepsilon' + \gamma(y'+a) - d + x'(-y'-a+\beta'))\, t \\ & + (\zeta' + \delta(y'+a) + \alpha d + (\gamma' + \alpha)x' + x'^2)\, t^2 \\ & + (\vartheta' + \varepsilon(y'+a) + \beta d + dy' + g + \\ & \quad x'(\delta' + \beta(y'+a) + y'^2 + ay' + b) + (a+1)x'^2)\, t^3 \\ & + (\lambda' + \zeta'(y'+a) + \gamma d + (\varepsilon' + \gamma(y'+a) - d)x' \\ & \qquad + (-y'-a+\beta')x'^2)\, t^4 \Big\} \\ & + \dots \end{aligned}$$

Nunc videndum est, ut potestates t^{-3} et t^0 amoveantur ex evolutione nostra, quod fit, si functioni nostrae additur:

$$\frac{(\alpha'+1)x+\delta'+\beta(y'+a)+y'^2+ay'+b+(\alpha'+1)x'}{G(x'y')_2}$$

$$=\frac{\delta'+a\beta+b+(a+\beta)y'+(\alpha'+1)\,(x'+x)+y'^2}{G(x'y')_2}$$

Tum erit

$$H(x\,y\,x'y')=\frac{G(x\,y\,y')}{(x'-x)\,G(x'y')_2}+\frac{(\alpha'+1)x+\delta'+\beta(y'+a)+y'^2+ay'+b+(\alpha'+1)x'}{G(x'y')_2}$$

et

$$4)\quad H(\bar{x}_t\,\bar{y}_t\,x'\,y')=H(x'\,y')_1\,t^{-4}+H(x'y')_2\,t^{-2}+H(x'\,y')_3\,t^{-1}+\sum_{1}^{\infty}{}_{\lambda}\,\overset{\lambda}{H}(x'\,y')\,t^{\lambda}\ ^{*)}$$

et quum illi numeri characteristici k_α sint: $k_1=4$, $k_2=2$, $k_3=1$ ponendum est:

$$H(x'y')_1=-\frac{1}{G(x'y')_2},\quad \overset{4}{H}(x'y')=-\frac{\lambda'+\zeta(y'+a)+\gamma d+(\varepsilon'+\gamma(y'+a)-d)x'+(-y'-a+\beta')x'^2}{G(x'y')_2}$$

$$H(x'y')_2=-\frac{\beta'-y'-a}{G(x'y')_2},\qquad \overset{2}{H}(x'y')=-\frac{\zeta'+\delta(y'+a)+\alpha d+(\gamma'+\alpha(y'+a))x'+x'^2}{G(x'y')_2}$$

$$H(x'y')_3=-\frac{\gamma'+\alpha(y'+a)+x'}{G(x'y')_2},\qquad \overset{1}{H}(x'y')=-\frac{\varepsilon'+\gamma(y'+a)-d+(-y'-a+\beta')x'}{G(x'y')_2}$$

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*) Quamquam non opus nobis est in iis, quae sequuntur, functione $H(xy\,x'y')$, liceat tamen afferre, quae forma ei tribui possit.

Est $$H(xy\,x'y')=\frac{1}{(x'-x)\,G(x'y')_2}\left\{\begin{array}{l}(y^2+yy'+y'^2)x+(y+y')(ax+d)+x^2+bx+g\\ +(\delta'+\beta a+b)(x'-x)+(\beta+a)y'(x'-x)+(\alpha'+1)(x'^2-x^2)+y'^2(x'-x)\end{array}\right\}$$

$$\delta'+\beta a+b=\frac{a^2-2b}{3}-\frac{a^2}{3}+b=\frac{b}{3},\quad \beta+a=-\frac{a}{3}+a=\frac{2a}{3},\quad \alpha'+1=-\frac{2}{3}+1=\frac{1}{3}$$

erit $$H(xy\,x'y')=\frac{1}{3(x'-x)\,G(x'y')_2}\left\{\begin{array}{r}3(y^2+yy'+y'^2)x+3(y+y')(ax+d)+3(x^2+bx+g)+b(x'-x)\\ +2ay'(x'-x)+(x'^2-x^2)+3y'^2(x'-x)\end{array}\right\}$$

$$=\frac{1}{3(x'-x)\,G(x'y')_2}\left\{\begin{array}{r}3y^2x+3y(ax+d)+3(x^2+bx+g)+3yy'x+3y'^2x+3y'(ax+d)\\ -x^2-bx\qquad\qquad +bx'+x'^2+2y'a(x'-x)\\ 3y'^2(x'-x)\end{array}\right\}$$

$$=\frac{1}{3(x'-x)\,G(x'y')_2}\left\{\begin{array}{r}3y^2x+3y(ax+d)+2(x^2+bx+g)+g+bx'+x'^2+2y'(ax'+d)\\ +3y'^2x'+3yy'x+y'(ax+d)\\ +3y^2x+y(ax+d)-3y^2x-y(ax+d)\end{array}\right\}$$

$$=\frac{1}{3(x'-x)\,G(x'y')_2}\left\{\begin{array}{r}6y^2x+4y(ax+d)+2(x^2+bx+g)+3y'^2x'+2y'(ax'+d)+x'^2+bx'+g\\ +(y'-y)\,(3xy+(ax+d))\end{array}\right\}$$

Ponatur nunc $$\frac{\partial G(xy)}{\partial y}=3y^2x+2y(ax+d)+x^2+bx+g=G(xy)_2$$

et $$\frac{\partial^2 G(xy)}{\partial y^2}=6yx+2(ax+d)=G(xy)_{22}$$

erit: $$H(xy\,x'y')=\frac{4G(xy)_2+(y'-y)\,G(xy)_{22}+2G(x'y')_2}{6(x'-x)\,G(x'y')_2}$$

vel etiam:

$$H(x'y')_1 = \frac{-1}{G(x'y')_2}, \quad H(x'y')_2 = \frac{(a-\beta')+y'}{G(x'y')_2}, \quad H(x'y')_3 = \frac{-(\gamma'+a\alpha)-\alpha y'-x'}{(G x'y')_2}$$

$$H'(x'y')_1 = -4\overset{4}{H}(x'y') = \frac{4}{G(x'y')_2}\left\{\lambda'+\zeta a+\gamma d+\zeta y'+(\varepsilon'+a\gamma-d)x'+\gamma y'x' + (\beta'-a)x'^2-y'x'^2\right\}$$

$$H'(x'y')_2 = -2\overset{2}{H}(x'y') = \frac{2}{G(x'y')_2}\left\{\zeta'+\delta a+\alpha a+\delta y'+(\gamma'+\alpha a)x'+\alpha x'y'+x'^2\right\}$$

$$H'(x'y')_3 = -\overset{1}{H}(x'y') = \frac{1}{G(x'y')_2}\left\{\varepsilon'+a\gamma-d+\gamma y'+(\beta'-a)x'-x'y'\right\}$$

et substitutis valoribus coefficientium

$$5)\quad H(x'y')_1 = \frac{-1}{G(x'y')_2}, \quad H(x'y')_2 = \frac{\frac{3a-1}{9}+y'}{G(x'y')_2}, \quad H(x'y')_3 = \frac{\frac{9a-2}{81}-\frac{1}{3}y'-x'}{G(x'y')_2}$$

$$H'(x'y')_1 = \frac{4}{G(x'y')_2}\left\{\frac{1}{59049}\left\{\begin{aligned}&243a^4-810a^3-2187a^2b+549a^2+3645ab+6561b^2-6561ac\\&-17496ad-132a-810b-4374c+1215d+13122g+11\end{aligned}\right\}\right.$$
$$+\frac{y'}{6551}\left\{-162a^3+162a^2-45a+729ab-243b-2187c+729d+4\right\}$$
$$+\frac{x'}{729}\left\{-27a^2+15a+162b-243d-2\right\}$$
$$\left.+\frac{x'y'}{81}(9a-1)+\frac{x'^2}{9}(1-3a)-y'x'^2\right\}$$

Est autem $G(x't\,y't)_2 = G(x'y')_2 + t\mathfrak{P}(t)$ si $x't = x'+t$ et $y't = y'+t\,\mathfrak{P}_1(t)$

et propter hoc $H(x't\,y't\;x'\,y) = \frac{1}{6(x'-x'_t)G(x'y')_2}\left\{6\,G(x'y')_2 + (x't-x')\overline{\mathfrak{P}}(x't-x')\right\}$

$$= \frac{1}{x'-x't} + \overline{\mathfrak{P}}_1(x't-x')$$

Ex hac forma functionis $H(x\,y\,x'\,y')$ facillime derivantur multae illae functiones, quae maximi in hac theoria sunt momenti. Exempli gratia habemus:

Si ponitur $H(\check{x}t\,\check{y}t\,x\,y) = \sum_\lambda \overset{\lambda}{H}(x\,y\,\check{x}\,\check{y})\,t^\lambda$ est $\overset{0}{H}(x\,y\,\check{x}\,\check{y}) = \frac{4\,G(\check{x}\,\check{y})_2 + (y-\check{y})\,G(\check{x}\,\check{y})_{22} + 2\,G(xy)_2}{6\,(x-\check{x})\,G(xy)_2}$

et, si $(\check{x}\,\check{y})$ est par singulare, pro quo igitur $G(\check{x}\,\check{y})_2 = 0$ sit, erit:

$$H(x\,y)xy = \frac{(y-yo)G(x\,y)_{22} + 2\,G(x\,y)_2}{6(x-\check{x})G(xy)_2} \quad \text{et } H(x\,y\,\check{x}\,\check{y}) = \frac{3\,G(x\,y)_{22} + (y-\check{y})\,G(\check{x}\,\check{y})_{222}}{6\,(x-\check{x})\,G(xy)_2} \quad \text{si } G_{222} = \frac{\partial^3 G}{\partial y^3}$$

$$H'(x'y')_2 = \frac{2}{G(x'y')_2} \left\{ \frac{1}{6561} (81a^3 - 162a^2 - 729ab + 63a + 486b + 4374c - 729d - 7) \right.$$
$$\left. + \frac{y'}{243} (-27a^2 + 9a + 81b - 1) + \frac{2-9a}{81} x' + \frac{1}{3} x'y' + x'^2 \right\}$$

$$H'(x'y')_3 = \frac{1}{G(x'y')_2} \left\{ \frac{1}{729} (-27a^2 + 15a + 162b - 243d - 2) \right.$$
$$\left. + \frac{9a-1}{81} y' + \frac{1-3a}{9} x' - x'y' \right\}$$

Harum functionum loco alias simpliciores inducere volumus. Ponatur primum:

$$H(xy)_1 = \frac{1}{G(xy)_2}, \quad \bar{H}(xy)_2 = \frac{y}{G(xy)_2}, \quad \bar{H}(xy)_3 = \frac{x}{G(xy)_3};$$

quae functiones autem pro functionibus $H'(xy)_\alpha$ ponendae sint, illa formula docet:

$$6) \quad \frac{d}{dx} H(xy\,x'y') - \frac{d}{dx'} H(x'y'xy) = \sum_{1}^{\varrho}{}_\alpha \left(H(x'y')_\alpha H'(xy)_\alpha - H(xy)_\alpha H'(x'y')_\alpha \right)$$

Si brevitatis causa ponitur:

Si autem ponitur $H(xy\,x_\tau y_\tau) \frac{d\bar{x}_\tau}{d_\tau} = -\sum_\mu H(xy\,x_0 y_0)_\mu \tau^\mu$ et $\overset{\vee}{x}\overset{\vee}{y}$ par est singulare, erit

$$H(xy\overset{\vee}{x}\overset{\vee}{y})_0 = \frac{(-4\,G(xy)_2 + (y - \overset{\vee}{y})\,G(xy)_{22})}{6(\overset{\vee}{x} - x)G(\overset{\vee}{x}\overset{\vee}{y})_1} \quad \text{ubi } G_1 = \frac{\partial G}{\partial x}$$

Simili modo, si $\bar{x}_\tau \bar{y}_\tau$ functionum par est, quod infinitos valores x, y nobis suppeditat, accipiemus:

$H(xy\,\bar{x}_0\,\bar{y}_0)_{-1} = 1$

$H(xy\,\bar{x}_0\,\bar{y}_0)_2 = x, \quad H(\bar{x}_t\,\bar{y}_t\,\bar{x}_0\,\bar{y}_0)_2 = t^{-3}$

$$H(xy\,\bar{x}_0\,\bar{y}_0)_4 = -\frac{3xy + (ax+d)}{3} = -\frac{1}{6} G(xy)_{22},$$

$$H(\bar{x}t\,\bar{y}t\,\bar{x}_0\,\bar{y}_0)_4 = t^{-5} - \frac{1}{3} t^{-4} + \frac{1-9a}{81\ 1} t^{-2} + \frac{28a^2 - 9a - 81b + 1}{243} t^{-1} + t\mathfrak{P}(t)$$

$H(xy\,\bar{x}_0\,\bar{y}_0)_5 = x^2, \quad H(\bar{x}_t\,\bar{y}_t\,\bar{x}_0\,\bar{y}_0)_5 = t^{-6}$

$$H(xy\,\bar{x}_0\,\bar{y}_0)_6 = \frac{36\,G(xy)_2 - (9y + 3a - 1)\,G(xy)_{22}}{54}$$
$$= \frac{36(3y^2x + 2y(ax+d) + x^2 + bx + g - (9y + 3a - 1)(6xy + 2ax + 2d)}{54}$$
$$= (y^2x + dy) + \frac{2}{3}x^2 + \frac{6a+1}{6}xy + \frac{-3a^2 + a + 18b}{27}x + \frac{-3ad + d + 36g}{54}$$

$$\text{et } H(\bar{x}_t\,\bar{y}_t\,\bar{x}_0\,\bar{y}_0)_6 = t^{-7} + \frac{-18a+5}{81} t^{-4} + \frac{-18a^2 + 9a + 54b - 81d - 1}{243} t^{-2}$$
$$+ \frac{324a^3 - 243a^2 + 99a - 1458ab - 4374ad + 729b\ \ 4374c - 729d - 10}{6561} t^{-1} + t\mathfrak{P}(t)$$

etc. etc. etc.

$$H(xy)_1 = \frac{\alpha}{G(xy)_2}, \qquad H(xy)_2 = \frac{\beta + \gamma y}{G(xy)_2}, \qquad H(xy)_3 = \frac{\delta + \varepsilon + \zeta x}{G(xy)_2},$$

$$H'(xy)_3 = \frac{\eta + \vartheta y + \varkappa x + \lambda x y}{G(xy)_2}$$

$$H'(xy)_2 = \frac{\mu + \nu y + \xi x + \pi x y, + \rho x^2}{G(xy)_2}, \quad H'(xy)_1 = \frac{\sigma + \tau y + \upsilon x + \phi x y + \psi x^2 + \omega x^2 y}{G(xy)_2}$$

erit:

$$H(x'y')_1\, H'(xy)_1 - H(xy)_1\, H'(x'y')_1$$
$$= \frac{1}{G(xy)_2\, G(x'y')_2} \left\{ \alpha\tau(y-y') + \alpha\upsilon(x-x') + \alpha\phi(xy - x'y') + \alpha\psi(x^2 - x'^2) + \alpha\omega(x^2 y - x'^2 y') \right\}$$

$$H(x'y')_2\, H'(xy)_2 - H(xy)_2\, H'(x'y')_2$$
$$= \frac{1}{G(xy)_2\, G(x'y')_2} \left\{ \begin{array}{l} \beta\nu(y-y') + \beta\xi(x-x') + \beta\pi(xy - x'y') + \beta\rho(x^2 - x'^2) \\ + \gamma\mu(y'-y) + \gamma\xi(y'x - yx') + \gamma\pi(xyy' - x'yy') \\ + \gamma\rho(x^2 y' - x'^2 y') \end{array} \right\}$$

$$H(x'y')_3\, H'(xy)_3 - H(xy)_3\, H'(x'y')_3$$
$$= \frac{1}{G(xy)_2\, G(x'y')_2} \left\{ \begin{array}{l} \delta\vartheta(y-y') + \delta k(x-x') + \delta\lambda(xy - x'y') \\ + \varepsilon\eta(y'-y) + \varepsilon k(xy' - x'y) + \varepsilon\lambda(xyy' - x'y'y) \\ + \zeta\eta(x'-x) + \zeta\vartheta(x'y - xy') + \zeta\lambda(xx'y - xx'y') \end{array} \right\}$$

Est igitur

$$\sum_{1}^{3}{}_{\alpha} \left(H(x'y')_\alpha\, H'(xy)_\alpha - H(xy)_\alpha\, H'(x'y')_\alpha \right)$$

$$= \frac{1}{G(xy)_2\, G(x'y')_2} \left\{ \begin{array}{l} A(y-y') + B(x-x') + C(xy - x'y') + D(x^2 - x'^2) + E(x^2 y - x'^2 y') \\ + F(xy' - x'y) + G(xyy' - x'y'y) + K(x^2 y' - x'^2 y) \\ + L(xyx' - x'yx) \end{array} \right\}$$

Quum alia ex parte hoc aequale esse debeat expressioni:

$$\left(\frac{1}{G(x'y')_2} \overline{H}'(xy)_1 - \frac{1}{G(xy)_2} \overline{H}'(x'y')_1 \right) + \left(\frac{y'}{G(x'y')_2} \overline{H}'(xy)_2 - \frac{y}{G(xy)_2} \overline{H}'(x'y')_2 \right)$$
$$+ \left(\frac{x'}{G(x'y')_2} H'(xy)_3 - \frac{x}{G(xy)_2} \overline{H}'(x'y')_3 \right)$$

videmus, ponendum esse:

$$H'(xy)_1 = \frac{Ay + Bx + Cxy + Dx^2 + Ex^2 y}{G(xy)_2}$$

$$\overline{H}'(xy)_2 = \frac{Fx + Gxy + Kx^2}{G(xy)_2}$$

$$\overline{H}'(xy)_3 = \frac{Lxy}{G(xy)_2}$$

Calculandi sunt coëfficientes $A, B, C, \ldots L$. Invenitur facile:

$$A = -\frac{1}{3}d \qquad B = \frac{3d-2b}{3}, \qquad C = -\frac{a}{3}, \qquad D = \frac{6a-2}{3} \qquad E = 4$$

$$F = O, \qquad G = 1, \qquad K = 2$$

$$L = 1$$

Erit ergo, si rursus $\mathrm{H}(xy)_\alpha$ et $\mathrm{H}'(xy)_\alpha$ pro $\overline{\mathrm{H}}(xy)_\alpha$ et $\overline{\mathrm{H}}'(xy)_\alpha$ scribuntur:

$$\mathrm{H}(xy)_1 = \frac{1}{G(xy)_2}, \mathrm{H}'(xy)_1 = \frac{4x^2y + \frac{6a-2}{3}x^2 - \frac{a}{3}xy + \frac{3d-2b}{3}x - \frac{d}{3}y}{G(xy)_2}$$

$$7) \qquad \mathrm{H}(xy)_2 = \frac{y}{G(xy)_2} \qquad \mathrm{H}'(xy)_2 = \frac{2x^2 + xy}{G(xy)_2}$$

$$\mathrm{H}(xy)_3 = \frac{x}{G(xy)_2} \qquad \mathrm{H}'(xy)_3 = \frac{xy}{G(xy)_2}$$

et reciproce:

$$\frac{x^2}{G(xy)_2} = \frac{1}{2}\mathrm{H}'(xy)_2 - \frac{1}{2}\mathrm{H}'(xy)_3$$

$$\frac{x^2y}{G(xy)_2} = \frac{1}{4}\mathrm{H}'(xy)_1 + \frac{1-3a}{12}\mathrm{H}'(xy)_2 + \frac{4a-1}{12}\mathrm{H}'(xy)_3$$

$$+ \frac{2b-3d}{12}\mathrm{H}(xy)_3 + \frac{d}{12}\mathrm{H}(xy)_2$$

Omnes functiones $\mathrm{H}(xy)_\alpha$ et $\mathrm{H}'(xy)_\alpha$ componuntur functionibus hujus formae: $\frac{x^\alpha y^\beta}{G(xy)_2}$, quam per $\mathrm{H}(xy)_{\alpha\beta}$ designare volumus. Itaque quod de integrali $\int \mathrm{H}(xy)_{\alpha\beta}\,dx$ valebit, valebit etiam de integralibus notris:

$$\int H(xy)_\alpha\,dx, \qquad \int H'(xy)_\alpha\,dx$$

Jam primum demonstremus, variationem integralis hujus: $\delta \int H(xy)_{\alpha\beta}\,dx$ per integralia primae ac secundae speciei et per functionem algebraicam exprimi posse.

Est autem:

$$8) \qquad \delta \int H(xy)_{\alpha\beta}\,dx = \int \left\{ \sum_{\alpha'\beta'} \frac{\partial \mathrm{H}_{\alpha\beta}}{\partial g_{\alpha'\beta'}} \delta g_{\alpha'\beta'} + \frac{\partial \mathrm{H}_{\alpha\beta}}{\partial y} \delta y \right\} \qquad \begin{pmatrix} \alpha' = 0, 1 \\ \beta' = 0, 1, 2, \end{pmatrix}$$

Quum autem sit:

$$G(xy) = xy^3 + y^2(ax + d) + y(x^2 + bx + g) + x^3 + cx + k = O$$

$$= xy^3 + g_{1,2}y^2x + g_{0,2}y^2 + yx^2 + g_{1\,1}yx + g_{0,1}y + x^3 + g_{1,0}x + g_{00} = O$$

$$G(xy)_2 = \frac{\partial G(xy)}{\partial y} = 3y^2x + 2y(g_{1,2}x + g_{0,2}) + x^2 + g_{11}x + g_{0,1}$$

$$\delta G(xy) = \sum_{\alpha'\beta'} \frac{\partial G(xy)}{\partial g_{\alpha'\beta'}} \delta g_{\alpha'\beta'} + \frac{\partial G(xy)}{\partial y} \delta y = O$$

$$\delta y = -\frac{1}{G(xy)_2} \sum_{\alpha'\beta'} \frac{\partial G(xy)}{\partial g_{\alpha'\beta'}} \delta g_{\alpha'\beta'}$$

$$g_{\alpha'\beta'} = \Big[G(xy)\Big]_{x^{\alpha'} y^{\beta'}} = \frac{1}{\beta'} \Big[G(xy)_2\Big]_{x^{\alpha'} y^{\beta'-1}}$$

$$\frac{\partial G(xy)}{\partial g_{\alpha'\beta'}} = x^{\alpha'} y^{\beta'}, \qquad \frac{\partial G(xy)_2}{\partial g_{\alpha'\beta'}} = \beta' x^{\alpha'} y^{\beta'-1}$$

erit

8a) $$\delta \int H(xy)_{\alpha\beta}\, dx = \sum_{\alpha'\beta'} \delta g_{\alpha'\beta'} \int \left\{ \frac{\partial H_{\alpha\beta}}{\partial g_{\alpha'\beta'}} - \frac{\partial H_{\alpha\beta}}{\partial y} \frac{\partial G(xy)}{\partial g_{\alpha'\beta'}} \frac{1}{G(xy)_2} \right\} dx$$

$$\begin{pmatrix} \alpha' = 0, 1 \\ \beta' = 0, 1, 2 \end{pmatrix}$$

$$\frac{\partial H(xy)_{\alpha\beta}}{\partial g_{\alpha'\beta'}} = \frac{\partial}{\partial g_{\alpha'\beta'}} \frac{x^\alpha y^\beta}{G(xy)_2} = -\frac{x^\alpha y^\beta}{G^2(xy)_2} \frac{\partial G(xy)_2}{\partial g_{\alpha'\beta'}} = -\frac{\beta' x^{\alpha+\alpha'} y^{\beta+\beta'-1}}{G^2(xy)_2}$$

$$\frac{\partial H(xy)_{\alpha\beta}}{\partial y} \frac{\partial G(xy)}{\partial g_{\alpha'\beta'}} = \frac{\beta x^\alpha y^{\beta-1} G(xy)_2 - x^\alpha y^\beta G(xy)_{22}}{G^2(xy)_2} x^{\alpha} y^{\beta'}$$

$$\text{ubi } G(xy)_{22} = \frac{\partial^2 G(xy)}{\partial y^2}$$

$$= x^{\alpha+\alpha'} y^{\beta+\beta'-1} \frac{\beta G(xy)_2 - y G(xy)_{22}}{G^2(xy)_2}$$

Tum erit

$$\frac{\partial H(xy)_{\alpha\beta}}{\partial g_{\alpha'\beta'}} - \frac{\partial H(xy)_{\alpha\beta}}{\partial y} \frac{\frac{\partial G(xy)}{\partial g_{\alpha'\beta'}}}{G(xy)_2} = -x^{\alpha+\alpha'} y^{\beta+\beta'-1} \frac{(\beta+\beta') G(xy)_2 - y G(xy)_{22}}{G^3(xy)_2}$$

et

9) $$\delta \int H(xy)_{\alpha\beta}\, dx = -\int \sum_{\alpha'\beta'} \frac{x^{\alpha+\alpha'} y^{\beta+\beta'-1} ((\beta+\beta') G(xy)_2 - y G(xy)_{22})}{G^3(xy)_2} \delta g_{\alpha'\beta'}\, dx$$

$$\begin{pmatrix} \alpha' = 0, 1 \\ \beta' = 0, 1, 2 \end{pmatrix}$$

Ut demonstretur, integrale hoc non esse tertiae speciei, sufficiet demonstrare functionem:

$$F(xy)_{m,n}\, dx = \frac{x^m y^{n-1} (n G(xy)_2 - y G(xy)_{22})}{G^3(xy)_2} dx$$

$$\text{ubi } m = 0, 1, 2, 3$$
$$n = 0, 1, 2, 3$$

in seriem evolutam non continere usquam potestatem $(x - x_\nu)^{-1}$, (si x_ν valor est, pro quo ipsa $F(xy)_{m,n}\, dx$ infinita fit.) Propterea necesse est evolutiones inquirere in valorum

x, y circuitu, pro quibus sit $G(xy)_2 = O$, deinde evolutionem pro $x = \infty$, $y = \infty$, et pro $x = O$, $y = \infty$.

Est autem primum, si $G(x_o y_o)_2 = O$, est si functiones sine argumento scriptae functiones argumentorum $(x_o y_o)$ designant:

$$G(xy) = G_1(x - x_o) + G_{22}\frac{(y-y_o)^2}{1.\ 2} + 2\,G_{21}\frac{(x-x_o)(y-y_o)}{1.\ 2} + G_{11}\frac{(x-x_o)^2}{1.\ 2} + \ldots = O,$$

tum erit

$$x - x_o = -\frac{1}{2}\frac{G_{22}}{G_1}(y-y_o)^2 + \left(\frac{1}{2}\frac{G_{22}G_{21}}{G_1^2} - \frac{G_{222}}{6\,G_1}\right)(y-y_o)^3 + \ldots$$

$$\text{vel} = c_2(y-y_o)^2 + c_3(y-y_o)^3 + c_4(y-y_o)^4 + \ldots$$

$$G(xy)_2 = G_{22}(y-y_o) + \frac{G_{222}\,G_1 - G_{21}\,G_{22}}{2\,G_1}(y-y_o)^2$$

$$+ \frac{1}{2}\left(c_3\,G_{21} - \frac{1}{2}\frac{G_{22}\,G_{212}}{G_1}\right)(y-y_o)^3 + \ldots$$

$$\text{vel} = m_1(y-y_o) + m_2(y-y_o)^2 + m_3(y-y_o)^3 + \ldots$$

$$\frac{1}{G(xy)_2} = \frac{1}{G_{22}(y-y_o)}\left\{1 - \frac{m_2}{G_{22}}(y-y_o) + \left(\frac{m_2^2}{m_1^2} - \frac{m_3}{m_1}\right)(y-y_o)^2 + \ldots\right\}$$

$$\frac{1}{G^3(xy)_2} = \frac{1}{G_{22}^3(y-y_o)^3}\left\{1 - 3\frac{m_2}{G_{22}}(y-y_o) + \frac{6m_2^2 - 3m_3\,m_1}{G_{22}^2}(y-y_o)^2 + \ldots\right\}$$

$$dx = -\frac{G_{22}}{G_1}(y-y_o)\,dy\left\{1 - \frac{3c_3\,G_1}{G_{22}}(y-y^o) - \ldots\right\}$$

$$\frac{dx}{G^3(xy)_2} = -\frac{dy}{G_2^2\,G_1(y-y_o)^2}\left\{1 - 3\frac{c_3\,G_1 + m_2}{G_{22}}(y-y_o) + \ldots\right\}$$

Simili modo est:

$$G(xy)_{22} = G_{22} + G_{222}(y-y_o) - \frac{1}{2}\frac{G_{22}\,G_{221}}{G_1}(y-y_o)^2 + \ldots$$

et etiam

$$G(xy)_2 = G_{22}(y-y_o) + \frac{1}{2}\left\{G_{222} - \frac{G_{21}\,G_{22}}{G_1}\right\}(y-y_o)^2 + \ldots$$

erit igitur

$$n\,G(xy)_2 - y\,G(xy)_{22} = n\,G(xy)_2 - (y_o + (y-y_o))\,G(xy)_{22}$$

$$= -G_{22}\left\{y_o + \frac{y_o G_{222} - (n-1)\,G_{22}}{G_{22}}(y-y_o) + \ldots\right\}$$

hoc multiplicandum est per:

$$\frac{dx}{G^3(xy)_2} = -\frac{dy}{G_{22}^2\,G_1(y-y_o)^2}\left\{1 - 3\frac{c_3\,G_1 + m_2}{G_{22}}(y-y_o) - \ldots\right\}$$

$$\frac{(n\,G(xy)_2 - y\,G(xy)_{22})\,dx}{G^3(xy)_1} = \frac{dy}{G_1\,G_{22}(y-y_o)^2}\left\{y_o - (n-1(y-y_o) + (y-y_o)^2\,\mathfrak{P}(y-y_o)\right\}$$

Est enim coëfficiens potestatis $(y-y_o)$:

$$\frac{1}{G_{22}}\left\{y_o G_{222}-(n-1)\ G_{22}-3y_o G_1 c_3-3y_o m_2\right\}$$

$$=-(n-1)+y_o\left\{G_{222}-\frac{3}{2}\frac{G_{21}G_{22}}{G_1}+\frac{G_{222}}{2}-\frac{3G_{222}}{2}+\frac{3}{2}\frac{G_{21}G_{22}}{G_1}\right\}$$

$$=-(n-1)$$

Restat factor $x^m y^{n-1}$.

$$x^m y^{n-1}=(x+(x-x_o))^m(y_o+(y-y_o))^{n-1}$$

$$=\left\{x_o-\frac{1}{2}\frac{G_{22}}{G_1}(y-y_o)^2+\dots\right\}^m\left\{y_o+(y-y_o)\right\}^{n-1}$$

$$=x_o^m y_o^{n-1}+x_o^m y_o^{n-2}(n-1)\ (y-y_o)+(y-y_o)^2\,\mathfrak{P}(y-y_o)$$

Habebimus ergo:

$$\frac{x^m y^{n-1}(nG(xy)_2-y\,G(xy)_{22})}{G^3(xy)_2}\,dx=\frac{dy}{G_{22}G_1(y-y_o)^2}\left\{\begin{array}{l}x_o^m y_o^n+x_o^m y_o^{n-1}(n-1)(y-y_o)+\\-x_o^m y_o^{n-1}(n-1)(y-y_o)+\dots\end{array}\right\}$$

$$=\frac{dy}{G_{22}G_1(y-y_o)^2}\left\{x_o^m y_o^n+(y-y_o)^2\,\mathfrak{P}(y-y_o)\right\}$$

Videmus, in serie pro $F(xy)_{m,n}\,dx$ non reperiri $(y-y_o)^{-1}$, si $G(x_o y_o)_2=O$, q. e. d.

Sie x prope nullitatis sumetur, habebimus:

$$y=-\frac{d}{x}+\frac{g-ad}{d}+\dots=-\frac{d}{x}\left\{1+\frac{ad-g}{d^2}x+\dots\right\}$$

$$G(xy)_2=\frac{d^2}{x}\left(1+\frac{2ad-3g}{d^2}x+x\mathfrak{P}(x)\right)$$

$$G(xy)_{22}=6xy+2(ax+d)=-4d+\frac{6g-4ad}{d}x+\dots$$

$$y\,G(xy)_{22}=\frac{4d^2}{x}+(8ad-10g)+\dots$$

$$\frac{1}{G(xy)_2}=\frac{x}{d}\left\{1+\frac{3g-2ad}{d^2}x+\dots\right\}$$

$$\frac{1}{G^3(xy)_2}=\frac{x^3}{d^2}\left\{1+\frac{9g-6ad}{d^2}x+\dots\right\}$$

$$nG(xy)_2-y\,G(xy)_{22}=\frac{d^2}{x}\left\{n-4+\frac{ad(2n-8)+g(10-3n)}{d^2}x+\dots\right\}$$

$$\frac{nG(xy)_2-y\,G(xy)_{22}}{G^3(xy)_2}=\frac{x^2}{d^4}\left\{n-4+\frac{ad(16-4n)+g(6n-26)}{d^2}x+\dots\right\}$$

$$y^{n-1}=(-1)^{n-1}\frac{d^{n-1}}{x^{n-1}}\left\{1+(n-1)\frac{ad-g}{d^2}x+\dots\right\}$$

$$\frac{x^m\, y^{n-1}\,(n\,G(x\,y)_2 - y\,G(x\,y)_{2\,2})}{G^3\,(x\,y)_2}$$
$$= (-1)^{n-1}\, d^{n-5}\, x^{3+m-n} \left\{ (n-5) + \frac{a\,d(n-4)\,(n-5) + g\,(n^2+n-22)}{d^2}\, x + \ldots \right\}$$

Quum m et n hos quatuor tantum valores accipiant 0, 1, 2, 3, exponens $(3+m-n)$ non fit negativus pro ullo valore m, n; quum igitur omnino negativae potestates in evolutione nostra non reperiantur, videmus integrale functionis nostrae etiam hoc loco non fieri logarithmicum. Jam statim concluditur ex theoremate de coëfficientibus logarithmicis, qui vocantur, et pro infinitis valoribus ipsius x et y functionem nostram in seriem evolutam non continere primam negativam argumenti potetatem. Unde fit, *ut integrale functionis $F(\mathrm{x\,y})_{m,n}$ non sit usquam logarithmicum.* Jam igitur ad *integralia primae ac secundae speciei* et functionem rationalem quantitatum x et y *reduci potest.*

Ponamus:

10). $$\int \frac{x^m\, y^{n-1}\,(n\,G\,(x\,y)_2 - y\;G(x\,y)_{2\,2})}{G^3\,(x\,y)_2}\,dx = \int \frac{\phi\,(x\,y)_{m,n}\,dx}{G\,(x\,y)_2} + \frac{\psi\,(x\,y)_{m,\,n}}{G\,(x\,y)_2}$$

ubi $\int \frac{\phi\,(x\,y)\,d\,x}{G\,(x\,y)_2}$ aggregatum designat omnium integralium primae ac secundae speciei cum coëfficientibus indeterminatis, $\psi(x\,y)$ autem functio est rationalis ipsorum x et y. Functioni ϕ haec tribui potest forma

$$\phi\,(x\,y)_{m,\,n} = l^{(1)}_{m,n} + l^{(2)}_{m,n}\,y + l^{(3)}_{m,\,n}\,x + l^{(4)}_{m,\,n}\,x\,y + l^{(5)}_{m,\,n}\,x^2 + l^{(6)}_{m,\,n}\,x^2\,y$$

Jam primum demonstrabitur functionem ψ esse functionem integram. Constat $\int \frac{\phi\;dx}{G(x\,y)_2}$ non fieri infinitum, nisi pro $\mathrm{x} = \infty$, $\mathrm{y} = \infty$. Contra fit:

$$\int F\,(x\,y)_{m,\,n}\,dx = \int \frac{\mathrm{x}^m\,\mathrm{y}^{n-1}\,(n\,G(x\,y)_2 - \mathrm{y}\;G\,(\mathrm{x\,y})_{2\,2})}{G^3\,(\mathrm{x\,y})_2}\;d\mathrm{x}$$

infinitum, si $G\,(\mathrm{x\,y})_2 = O$. Reperimus enim, si $\mathrm{x}_o\,\mathrm{y}_o$ par est singulare, pro quo $G\,(\mathrm{x}_o\,\mathrm{y}_o)_2 = O$.

$$\int F(\mathrm{x\,y})_{mn}\;d\mathrm{x} = \int \frac{d\,\mathrm{y}}{G(\mathrm{x}_o\,\mathrm{y}_o)_{2\,2}\,G(\mathrm{x}_o\,\mathrm{y}_o)_1}\;\frac{1}{(\mathrm{y}-\mathrm{y}_o)^2}\left\{\mathrm{x}_o^m\,\mathrm{y}_o^n + (\mathrm{y}-\mathrm{y}_o)^2\,\mathfrak{P}\,(\mathrm{x}-\mathrm{x}_o)\right\}$$
$$= \frac{-\,\mathrm{x}_o^m\,\mathrm{y}_o^n}{G(\mathrm{x}_o\,\mathrm{y}_o)_{2\,2}\,G(\mathrm{x}_o\,\mathrm{y}_o)_1}\,(\mathrm{y}-\mathrm{y}_o)^{-1} + (\mathrm{y}-\mathrm{y}_o)\,\overline{\mathfrak{P}}\,(\mathrm{y}-\mathrm{y}_o)$$

Potestates negativa tollitur, si per $G\,(\mathrm{x\,y})_2$ utramque aequationis hujus partem multiplicamus; est enim:

$$G(xy)_2 = G(x_o y_o)_{22}(y-y_o) + \ldots .$$

erit propterea

$$G(xy)_2 \int F(xy)_{m,n}\, dx = -\frac{x_o^m y_o^n}{G(x_o y_o)_1} + (y-y_o)\,\overline{\overline{\mathfrak{P}}}\,(y-y_o)$$

Quum autem sit:

$$G(xy)_2 \left\{ \int F(xy)_{m,n}\, dx = -\int \frac{\phi(xy)_{m,n}}{G(xy)_2} dx \right\} = \psi(xy)$$

videmus, $\psi(xy)$ non fieri infinitum, dum $G(xy)_2$ evanescit.

Si utimur porro evolutionibus functionis $F(xy)_{m,n}$ pro $x=0$, $y=\infty$, jam prius repertis, ubi negativas potestates omnino non reperiri vidimus, habebimus:

$$\int F(xy)_{m,n}\, dx = (-1)^{n-1} d^{n-5} \int \left\{ (n-4)\, x^{3+m-n} + \text{potestates sublimiores} \right\} dx$$

$$= (-1)^{n-1} d^{n-5} \left\{ \frac{(n-4)\, y^{4+m-n}}{4+m-n} + \text{pot. sublm.} \right\}$$

$$= x^{4+m-n} \{ \text{const.} + x\,\mathfrak{P}(x) \}$$

Alia ex parte habemus

$$G(xy)_2 = \frac{d^2}{x} \left(1 + \frac{2ad-3g}{d^2} x + x^2\,\mathfrak{P}(x) \right)$$

erit igitur:

$$G(xy)_2 \int F(xy)_{m,n}\, dx = x^{3+m-n} \{ \text{Const.} + x\,\mathfrak{P}(x) \}$$

non reperientur igitur hic potestates negativae, quum jam prius viderimus, esse $3 + m - n \geqq 0$.

Erit praeterea pro iisdem parvis valoribus x:

$$\frac{1}{G(xy)_2} = \frac{x}{d} \left\{ 1 + \frac{3g-2ad}{d^2} x + \ldots \right\}$$

$$\frac{y}{G(xy)_2} = -\left\{ 1 + \frac{2g-ad}{d^2} x + \ldots \right\}$$

$$\frac{\phi(xy)_{m,n}}{G(xy)_2} = \frac{y\left(l^{(6)}_{m,n} x^2 + l^{4}_{m,n} x + l^{(2)}_{m,n} \right) + l^{(5)}_{m,n} x^2 + l^{(3)}_{m,n} x + l^{(1)}_{m,n}}{G(xy)_2}$$

$$= \left\{ -l^{(2)}_{m,n} + x\,\mathfrak{P}(x) \right\}$$

$$\int \frac{\phi(xy)_{m,n}}{G(xy)_2} dx = x\,\mathfrak{P}(x) \text{ et } G(xy)_2 \int \frac{\phi(xy)\, dx}{G(xy)_2} = \mathfrak{P}(x)$$

(Post integrationem factam non additur „constans". Propter aequationem enim

$$\int F(xy)_{m,n}\, dx = \int \frac{\phi(xy)_{m.n}}{G(xy)_2} + \frac{\psi(xy)_{m,n}}{G(xy)_2}$$

supponitur illa functioni $\psi(xy)_{m,n}$ inesse.)

Tum erit

$G(\mathrm{x\,y})_2 \int \left(F(\mathrm{x\,y})_{m,n} - \frac{\phi(\mathrm{x\,y})_{m,n}}{G(\mathrm{x\,y})_2} \right) dx = \overline{\overline{\mathfrak{P}}}(\mathrm{x})$, ubi $\overline{\overline{\mathfrak{P}}}(\mathrm{x})$ seriem infinitam designat, quae positivas tantum continet potestates. Videmus, etiam si x nullitati prope jacet, esse

$$\psi_{m,n}(\mathrm{x\,y}) = \overline{\overline{\mathfrak{P}}}(\mathrm{x}),$$

unde concluditur, functionem $\psi_{m,n}(\mathrm{x\,y})$ non fieri infinitam, nisi si $\mathrm{x} = \infty$, et $\mathrm{y} = \infty$. Propter hanc proprietatem componitur $\psi(\mathrm{x\,y})_{m,n}$ ex illis functionibus ξ_3, ξ_5, ξ_7 (per additionem et multiplicationem), de quibus pag. 61, 62 tractavimus. Est igitur

11) $\psi(\mathrm{x\,y})_{m,n} = P(\mathrm{x})_{m,n}(\mathrm{x\,y}^2 + d\mathrm{y}) + Q(\mathrm{x})_{m,n}\,\mathrm{x\,y} + R(\mathrm{x})_{m,n}$

ubi P, Q, R functiones integrae sunt solius x. Designemus gradus ipsarum resp. per p, q, r. Numeri hi facile destinabuntur, si utramque partem aequationis:

12) $$\int F(\mathrm{x\,y})_{m,n}\, d\mathrm{x} - \int \frac{\phi(\mathrm{x\,y})_{m,n}\, d\mathrm{x}}{G(\mathrm{x\,y})_2} = \frac{\psi(\mathrm{x\,y})_{m,n}}{G(\mathrm{x\,y})_2}$$

pro magnis valoribus x et y evolvemus in seriam infinitam.

Liceat hic omnes evolutiones scribere, quibus in iis, quae sequuntur, utemur. Reperiemus:

$$G(\bar{\mathrm{x}}_t\, \bar{\mathrm{y}}_t)_2 = 3\,t^{-7} \left\{ 1 - \frac{1}{3}t + \frac{1}{9}t^2 + \frac{2-18a}{81}t^3 + \frac{a^2-3b}{9}t^4 + \frac{-54a^2+18a+162b-2}{729}t^5 + \frac{-324a^3-243a^2-1458ab+72a+486b+4374c-1458d-7}{6561}t^6 + \frac{2ad-3g}{9}t^7 + \left\{ \frac{729a^4-1296a^3-4374a^2b+684a^2+4374ab}{59049} \frac{+6561b^2-8748ad-144a-810b-4374c}{59049} + \frac{+1458d+13122g+11}{59049} \right\} t^8 + \ldots\ldots \right\}$$

$$\frac{1}{G(\bar{\mathrm{x}}_t\, \bar{\mathrm{y}}_t)_2} = \frac{t^7}{3} \left\{ 1 + \frac{1}{3}t + \frac{18a-5}{81}t^3 + \frac{-27a^2+36a+81b-7}{243}t^4 + \frac{-324a^3+810a^2+1458ab-360a-1215b-4374c+1458d+44}{6561}t^6 + \left\{ \frac{-1620a^3+1890a^2+5832ab-4374ad-630a-2268b-8748c}{19683} + \frac{2916d+6561g+65}{19683} \right\} t^7 + t^9\,\mathfrak{P}(t) \right\}$$

$$\frac{1}{G^3(\bar{x}_t\,\bar{y}_t)_2} = \frac{t^{21}}{27}\Big\{1 + t + \frac{1}{3}t^2 + \frac{18a-4}{27}t^3 + \frac{-27a^2+72a+81b-17}{81}t^4 + \frac{-54a^2+90a+162b-19}{243}t^5 + \frac{-108a^3+351a^2+486ab-144a-324b-1458c+486d+16}{729}t^6 + \dots\Big\}$$

$$\frac{\bar{y}_t}{G(\bar{x}_t\,\bar{y}_t)_2} = \frac{t^5}{3}\Big\{-1 + \frac{1-3a}{9}t^2 + \frac{4-18a}{81}t^3 + \frac{-108a^2+60a+162b-243d-8}{729}t^5 + \frac{405a^3-810a^2-1458ab+315a+729b+2187c-1458d-35}{6561}t^6 + \frac{1}{59049}\Big(1701a^4-5676a^3+3780a^2-8748a^2b+11664ab + 6561b^2+13122ac-17496ad-924a-2268b -10935c+486d+13122g+77\Big)t^8 + \dots\Big\}$$

$$\frac{\bar{y}_t^{\,2}}{G(\bar{x}_t\,\bar{y}_t)_2} = \frac{t^3}{3}\Big\{1 - \frac{1}{3}t + \frac{6a-1}{9}t^2 + \frac{54a^2-36a-81b+5}{243}t^4 + \dots\Big\}$$

Praeterea est $\bar{x}_t = t^{-3}$, $\bar{x}_t^2 = t^{-6}$, $\qquad \dfrac{d\bar{x}_t}{dt} = -3t^{-4}$

$$\bar{y}_t\,G(\bar{x}_t\,\bar{y}_t)_{22} = t^{-7}\Big\{6 - 4t + \frac{6a+2}{3}t^2 + \frac{4-54a}{27}t^3 + \frac{4a^2-12b}{3}t^4 + \frac{-162a^3+42a+324b+486d-4}{243}t^5 + \Big(\frac{1620a^3-1296a^2-5832ab+324a+2430b}{2187} + \frac{8748c-4374d-32}{2187}\Big)t^6 + \dots\Big\}$$

$$n\,G(\bar{x}_t\,\bar{y}_t)_2 - \bar{y}_t\,G(\bar{x}_t\,\bar{y}_t)_{22} = t^{-7}\Big\{(3n-6) + (4-n)t + \frac{n-2-6a}{3}t^2 + \dots\Big\}$$

$$\bar{y}_t^{\,n-1} = (-1)^{n-1}\,t^{-2(n-1)}\Big\{1 - \frac{n-1}{3}t + \dots\Big\}$$

$$\bar{x}_t^m \bar{y}_t^{n-1} (n G(\bar{x}_t \bar{y}_t)_2 - y_t G(\bar{x}_t \bar{y})_{22}) = (-1)^{n-1}) \; t^{-5-2n-3m} \{(3n-6) + (2+2n-n^2)\, t + \dots\}$$

$$3 G_2 - y G_{22} = t^{-7} \left\{ 3 + t + \frac{1-6a}{3} t^2 + \frac{2}{27} t^3 + \frac{-a^2+3b}{3} t^4 + \frac{12a+162b-486d-2}{243} t^5 + \frac{-648a^3-567a^2+1458ab-108a-972b-4374c+11}{2187} t^6 + \dots \right\}$$

$$2 G_2 - y G_{22} = t^{-6} \left\{ 2 - 2at + \frac{2a}{3} t^2 + \frac{-2a^2+6b}{3} t^3 + \dots \right\}$$

$$G_2 - y G_{22} = t^{-7} \left\{ -3 + 3t - \frac{1+6a}{3} t^2 + \dots \right\}$$

$$-G_{22} = t^{-5} (6 + t \mathfrak{P}(t))$$

$$y^2 (3 G_2 - y G_{22}) = t^{-11} \left\{ 3 - t + \frac{18+1}{27} t^3 + \frac{-54a^2-36a-81b+1}{81} t^4 + a^2 t^5 + \frac{-1782a^3-891a^2+2916ab-90a-1701b+1458d+4}{2187} t^6 + \dots \right\}$$

$$y (2 G_2 - y G_{22}) = t^{-8} \left\{ -2 + \frac{6a+2}{3} t - 2a\, t^2 + \frac{108a^2+36a-162b-2}{81} t^3 + \dots \right\}$$

$$\frac{\bar{x}_t^m \bar{y}_t^{n-1} (n G(\bar{x}_t \bar{y}_t)_2 - \bar{y}_t G(\bar{x}_t \bar{y}_t)_{22}}{G^3 (x_t y_t)_2} = \frac{(-1)^{n-1} t^{16-(2n+3m)}}{27} \{3(n-2) + (4-n)(n-1)\, t + \dots\}$$

$$F(xy)_{m,n} \frac{d\bar{x}_t}{dt} = \frac{\bar{x}_t^m \bar{y}^{n-1} (n G(\bar{x}_t \bar{y}_t)_2 - \bar{y}_t G(\bar{x}_t \bar{y}_t)_{22})}{G^3 (\bar{x}_t \bar{y}_t)_2} \frac{d\bar{x}_t}{dt}$$
$$= \frac{(-1)^n t^{12-(2n+3m)}}{9} \{3(n-2) + (4-n)(n-1)\, t + \dots\} \;{}^{*)}$$
$$= (m,n)_{-3}\, t^{-3} + (m,n)_{-2}\, t^{-2} + (m,n)_0 + (m,n)_1\, t + (m.n)_3\, t^3 + \dots$$

Quum $2n + 3m$ pro maximo valore „15" habeat, videmus $F(\bar{x}_t \bar{y}_t)\, m, n\, \frac{d\bar{x}_t}{dt}$ evolutam in seriem non continere negativas potestates variabilis t ultra tertiam. Praeterea est $(m,n)_2 = 0$, ut sequentes evolutiones speciales demonstrant.

Est enim:

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*) Evolutio haec non continet unquam: t^{-1}, ut jam prius demonstravimus. Sit enim $12-(2n+3m) = -1$, erit $3m = 13 - 2n$, et quum m numerus integer sit ponendum est $n=2$; tum autem evanescit primus coëfficiens $3(n-2)$, et potestas t^{-1} per se ex serie excedit.

$$\frac{\bar{y}_t^2\,(3\,G(\bar{x}_t\,\bar{y}_t)_2 - \bar{y}_t\,G(\bar{x}_t\bar{y}_t)_{22})}{G^3\,(\bar{x}_t\,\bar{y}_t)_2} = \frac{t^{10}}{27}\Big\{3 + 2\,t + \frac{72\,a-20}{27}\,t^3 + \frac{-135\,a^2+180\,a+162\,b-35}{81}\,t^4 + \left.\begin{matrix}\dfrac{2754\,a^3+5427\,a^2+7290\,ab-2682\,a-6804\,b}{2187} \\ + \dfrac{-13122c+5832d+316}{2187}\end{matrix}\right\}t^6 + \ldots\Big\}$$

$$\frac{\bar{y}_t(2\,G_2 - \bar{y}_t\,G_{22})}{G^3\,(\bar{x}_t\,\bar{y}_t)_2} = \frac{t^{13}}{27}\Big\{-2 + \frac{6\,a-4}{3}\,t + \frac{108\,a^2-180\,a-162\,b+28}{81}\,t^3 + \ldots\Big\}$$

$$\frac{G\,(\bar{x}_t\,\bar{y}_t)_2 - \bar{y}_t\,G\,(\bar{x}_t\,\bar{y}_t)_{22}}{G^3\,(\bar{x}_t\,\bar{y}_t)_2} = \frac{t^{14}}{27}\Big\{-3 + \frac{7-6\,a}{3}\,t^2 + \ldots\Big\}$$

$$\frac{-G\,(\bar{x}_t\,\bar{y}_t)_{22}}{G^3\,(\bar{x}\,\bar{y}_t)_2} = \frac{t^{16}}{27}\Big\{6 + t\,\mathfrak{P}\,(t)\Big\}$$

Erit ergo:

$$F\,(\bar{x}_t\,\bar{y}_t)_{3,3}\,\frac{d\,\bar{x}_t}{dt} = \frac{\bar{x}_t^3\;\bar{y}_t^2\,\{3\;G\,(\bar{x}_t\,\bar{y}_t)_2 - \bar{y}_t\,G(\bar{x}_t\,\bar{y}_t)_{22}\}}{G^3\,(\bar{x}_t\,\bar{y}_t)_2}\,\frac{d\,\bar{x}_t}{dt}$$

13)
$$= -\frac{1}{3}\,t^{-3} - \frac{2}{9}\,t^{-2} + \frac{-72\,a+20}{243} + \frac{135\,a^2-180\,a-162\,b+35}{729}\,t + \frac{1}{19683}\left\{\begin{matrix}-2574a^3-5427a^2-7290ab+2682ab+6804b \\ +13122\,c-5232\,d-316\end{matrix}\right\}t^3 + \ldots$$

$$F\,(\bar{x}_t\,\bar{y}_t)_{2,3}\,\frac{d\,\bar{x}_t}{dt} = -\frac{1}{3} - \frac{2}{9}\,t + \frac{-72\,a+20}{27}\,t^3 + \ldots$$

$$F\,(\bar{x}_t\,\bar{y}_t)_{1,3}\,\frac{d\,\bar{x}_t}{dt} = -\frac{1}{3}\,t^3 - \frac{2}{9}\,t^4 + \ldots$$

$$F\,(\bar{x}_t\,\bar{y}_t)_{0,3}\,\frac{d\,\bar{x}_t}{dt} = -\frac{1}{3}\,t^6 - \frac{2}{9}\,t^7 + \ldots$$

$$F\,(\bar{x}_t\,\bar{y}_t)_{3,2}\,\frac{d\,\bar{x}_t}{dt} = \frac{2}{9} + \frac{-6a+4}{27}\,t + \frac{-108\,a^2+180\,a+162\,b-28}{729}\,t^3 + \ldots$$

$$F\,(\bar{x}_t\,\bar{y}_t)_{2,2}\,\frac{d\,\bar{x}_t}{dt} = \frac{2}{9}\,t^3 + \frac{-6\,a+4}{27}\,t^4 + \ldots$$

$$F\,(\bar{x}_t\,\bar{y}_t)_{1,2}\,\frac{d\,\bar{x}_t}{dt} = \frac{2}{9}\,t^6 + \frac{-6\,a+4}{28}\,t^7 + \ldots$$

$$F\,(\bar{x}_t\,y_t)_{0,2}\,\frac{d\,\bar{x}_t}{dt} = \frac{2}{9}\,t^9 + \ldots$$

$$F(\bar{x}_t\,\bar{y}_t)_{3,1}\,\frac{d\bar{x}_t}{dt} = \frac{1}{3}t + \frac{6a-5}{27}t^3 + \ldots \qquad F(\bar{x}_t\,\bar{y}_t)_{3,0}\,\frac{d\bar{x}_t}{dt} = -\frac{2}{3}t^3 + \ldots\ldots$$

$$F(\bar{x}_t\,\bar{y}_t)_{2,1}\,\frac{d\bar{x}_t}{dt} = \frac{1}{3}t^4 + \ldots \qquad F(\bar{x}_t\,\bar{y}_t)_{2,0}\,\frac{d\bar{x}_t}{dt} = -\frac{2}{3}t^6 + \ldots\ldots$$

$$F(\bar{x}_t\,\bar{y}_t)_{1,1}\,\frac{d\bar{x}_t}{dt} = \frac{1}{3}t^7 + \ldots \qquad F(\bar{x}_t\,\bar{y}_t)_{1,0}\,\frac{d\bar{x}_t}{dt} = -\frac{2}{3}t^9 + \ldots$$

$$F(\bar{x}_t\,\bar{y}_t)_{0,1}\,\frac{d\bar{x}_t}{dt} = \frac{1}{3}t^{10} + \ldots \qquad F(\bar{x}_t\,\bar{y}_t)_{0,0}\,\frac{d\bar{x}_t}{dt} = -\frac{2}{3}t^{12} + \ldots\ldots$$

Revertamur ad aequationem nostram 12):

$$\int F(x\,y)_{m,n}\,dx - \int \frac{\phi(x\,y)_{m,n}}{G(x\,y)_2}\,dx = \frac{\psi(x\,y)_{m,n}}{G(x\,y)_2}$$

14) vel $F(x\,y)_{m,n} - \dfrac{\phi(x\,y)_{m,n}}{G(x\,y)_2} = \dfrac{d}{dx}\,\dfrac{\psi(x\,y)_{m,n}}{G(x\,y)_2}$

Aequatio haec fit identica, si differentiatione facta, pro x, y functionum par aliquod $x_t\,y_t$ substituitur, quod aequationi illi $G(x\,y) = 0$ satisfacit. Substituamus $\bar{x}_t\,\bar{y}_t$ erit:

$$\frac{\phi(\bar{x}_t\,\bar{y}_t)_{m,n}}{G(\bar{x}_t\,\bar{y}_t)_2}\,\frac{d\bar{x}_t}{dt} = \frac{l^{(6)}\,\bar{x}_t^2\,\bar{y}_t + l^{(5)}\,\bar{x}_t^2 + l^{(4)}\,\bar{x}_t\,\bar{y}_t + l^{(3)}\,\bar{x}_t + l^{(2)}\,\bar{y}_t + l^{(1)}}{G(\bar{x}_t\,\bar{y}_t)_2}\,\frac{d\bar{x}_t}{dt}$$

$$= l^{(6)}_{m,n}\,t^{-5} + \left(l^{(6)}\,\frac{3a-1}{9} - m_5\,l^{(5)}\right)t^{-3}$$

$$+ \left(l^{(6)}_{m,n}\,\frac{18a-4}{81} - \frac{1}{3}\,l^{(5)}_{m,n} + l^{(4)}_{m,n}\right)t^{-2} + \mathfrak{P}(t)$$

$$\frac{\psi(\bar{x}_t\,\bar{y}_t)_{m,n}}{G(\bar{x}_t\,\bar{y}_t)_2} = \frac{P(\bar{x}_t)_{m,n}\,(xy^2 + dy) + Q(\bar{x}_t)_{m,n}\,\bar{x}_t\,\bar{y}_t + R(\bar{x}_t)_{m,n}}{G(\bar{x}_t\,\bar{y}_t)_2}$$

$$= \frac{1}{G(\bar{x}_t\,\bar{y}_t)_2}\left\{(p_0\,\bar{x}_t^p + p_1\,\bar{x}_t^{p-1} + \ldots)\;(\bar{x}_t\,\bar{y}_t^2 + d\,\bar{y}_t) + (q_0\,\bar{x}_t^q + q_1\,\bar{x}_t^{q-1} + \ldots)\bar{x}_t\,\bar{y}_t + r_0\,\bar{x}_t^r + r_1\,\bar{x}_t^{r-1} + \ldots + r_r\right\}$$

$$\bar{x}_t \bar{y}_t^2 + d\bar{y}_t = t^{-7}\,\mathfrak{P}_1(t), \quad \bar{x}_t \bar{y}_t = t^{-5}\,\mathfrak{P}_2(t), \quad \frac{1}{G(\bar{x}_t \bar{y}_t)_2} = t^7\,\mathfrak{P}(t)$$

erit igitur:

$$\frac{\psi(\bar{x}_t \bar{y})_{m,n}}{G(\bar{x}_t \bar{y}_t)_2} = (c_1 t^{-3p} + \ldots) + (c_2 t^{2-3q} + \ldots) + (c_3 t^{7-3r} + \ldots)$$

$$\left(\frac{d}{dx}\,\frac{\psi(\bar{x}_t \bar{y}_t)_{m,n}}{G(\bar{x}_t \bar{y}_t)_2}\right)\frac{d\bar{x}_t}{dt} = \frac{d}{dt}\,\frac{\psi(\bar{x}_t \bar{y}_t)_{m,n}}{G(\bar{x}_t \bar{y}_t)_2}$$

$$= (\bar{c}_1 t^{-3p-1} + \ldots) + (\bar{c}_2 t^{1-3q} + \ldots) + (\bar{c}_3 t^{6-3r} + \ldots$$

Fac substitutas esse has series in

$$F(\bar{x}_t \bar{y}_t)_{m,n}\,\frac{d\bar{x}_t}{dt} - \frac{\phi(\bar{x}_t \bar{y}_t)_{m,n}}{G(\bar{x}_t \bar{y}_t)_2}\,\frac{d\bar{x}_t}{dt} = \frac{d}{dt}\,\frac{\psi(\bar{x}_t \bar{y}_t)}{G(\bar{x}_t \bar{y}_t)_2}$$

Videmus, summam negativam potestatem, quae in evolutione $F(x\,y)_{m,n}\,\frac{d\bar{x}_t}{dt}$ reperiri possit, esse t^{-3}, contra continet $\frac{\phi(\bar{x}_t \bar{y}_t)\frac{d\bar{x}_t}{dt}}{G(\bar{x}_t \bar{y}_t)_2}$ evoluta semper hanc negativam potestatem: t^{-5}. Jam necesse est, ut eadam potestates, in dextra etiam parte aequationis nostrae reperiatur, quae conditio has nobis suppeditat relationes, ad gradus p, q, r determinandos:

$$-5 \leqq -3p-1$$

$$-5 \leqq -3q+1$$

$$-5 \leqq -3r+a$$

Quum una harum relationum vera aequatio esse debeat, et p, q, r numeri sint integri et positivi, habemus necessario

$$-5 = -3q+1$$

id est $\quad 3q = 6$

et $\quad q = 2$

Praeterea est $\quad 3p < 4$

$\quad 3r < 11$

ponendum igitur est $p = 1$, $r = 3$.

Sit igitur

$$P(x)_{m,n} = p_{m,n}^{(1)}\,x + p_{m,n}^{(0)}$$

$$\mathrm{x}\, Q\,(\mathrm{x})_{m,n} = q^{(3)}_{m,n}\ \mathrm{x}^3 + q^{(2)}_{m,n}\ \mathrm{x}^2 + q^{(1)}_{m,n}\ x$$

$$R\,(\mathrm{x})_{m,n} = r^{(3)}_{m,n}\ \mathrm{x}^3 + r^{(2)}_{m,n}\ \mathrm{x}^2 + r^{(1)}_{m,n}\,\mathrm{x} + r^{(1)}_{m,n}$$

Brevitatis causa scribemus:

$$p_\alpha \text{ pro } p^{\alpha}_{m,n} \quad \text{etc.}$$

Nil restat nunc, nisi ut demonstremus, quomodo indeterminati adhuc coëfficientes $l^{(1)}_{m,n}, \ldots\ l^{(6)}_{m,n}$, $p_\alpha, q_\alpha, r_\alpha$, sint calculandi. Jam aequatio identica, quam modo tractavimus, comparatis coëfficientibus utriusque partis, aequationes lineares ad hos coëfficientes determinandos nobis suppeditat. Sed methodus haec hac difficultate laborat, quod sit permolestum, tot evolutionum membra calculare, quot nobis opus sunt. Tum autem intelligi non possit, quae sit natura denominatoris, omnibus quantitatibus l, p, q, r communis, quod autem summi est momenti. Evolutione hac pro $\mathrm{x} = \infty$, $\mathrm{y} = \infty$ tantum utimur, ut coëfficientes $l^{(\alpha)}_{m,n}$ . . ., per coëfficientes p, q, r, exprimantur. Ad hos coëfficientes autem determinandos alio modo aequationum linearium systema derivaturi sumus.

Contemplemur aequationem nostram:

$$\frac{\mathrm{x}^m\,\mathrm{y}^{n-1}\,(n\,G(\mathrm{x}\,\mathrm{y})_2 - \mathrm{y}\,G(\mathrm{x}\,\mathrm{y})_{2\,2})}{G^3\,(x\,\mathrm{y})_2} = \frac{\phi\,(x\,\mathrm{y})_{m,n}}{G\,(x\,\mathrm{y})_2} + \frac{d}{d\,x}\,\frac{\psi\,(x\,\mathrm{x})_{m,n}}{G\,(x\,\mathrm{y})_2}$$

Sit $\dfrac{\partial\,\psi\,(x\,\mathrm{y})}{\partial\,x} = \psi\,(x\,\mathrm{y})_1$, $\dfrac{\partial\,\psi\,(x\,\mathrm{y})}{\partial\,\mathrm{y}} = \psi\,(x\,\mathrm{y})_2$; differentiatione facta obtinemus, quum sit

$$\frac{d\mathrm{y}}{dx} = -\,\frac{G\,(x\,\mathrm{y})_1}{G\,(x\,\mathrm{y})_2}:$$

$$\frac{d}{d\,x}\,\frac{\psi\,(x\,\mathrm{y})}{G\,(x\,\mathrm{y})_2} = \frac{G(x\mathrm{y})_2\,\psi(x\mathrm{y})_1 - \psi(x\mathrm{y})G(x\mathrm{y})_{2,1}}{G^2\,(x\mathrm{y})_2} - \frac{G\,(x\,\mathrm{y})_1}{G(x\,\mathrm{y})_2}\,\frac{G(x\mathrm{y})_2\psi(x\mathrm{y})_2 - \psi(x\mathrm{y})G(x\mathrm{y})_{2\,2}}{G^2\,(x\,\mathrm{y})_2}$$

$$= \frac{\psi(x\mathrm{y})_1\,G^2(x\mathrm{y})_2 - G(x\mathrm{y})_2(\psi(x\,\mathrm{y})G(x\mathrm{y})_{2,1} + \psi(x\mathrm{y})_2\,G(x\mathrm{y})_1) + \psi(x\mathrm{y})G(x\mathrm{y})_1\,G(x\,\mathrm{y})_{2\,2}}{G^3\,(x\,\mathrm{y})_2}$$

quod brevitatis causa etiam hoc modo scribi potest:

$$\frac{d}{d\,x}\,\frac{\psi}{G_2} = \frac{\psi_1\,G_2^2 - G_2\,(\psi\,G_{2,1} + \psi_2\,G_1) + \psi\,G_1\,G_{2\,2}}{G_2^3}$$

Indicibus 1, 2 differentiationem partialem secundum resp. x, y designamus.

Erit igitur:

$$\frac{x^m\,\mathrm{y}^{n-1}\,(n\,G\,(x\,\mathrm{y})_2 - \mathrm{y}\,G\,(x\,\mathrm{y})_{2\,2})}{G^3\,(\mathrm{x}\,\mathrm{y})_2} = \frac{(\phi + \psi_1)\,G_2^2 - G_2\,(\psi\,G_{2\,1} + \psi_2\,G_1) + \psi\,G_1\,G_{2\,2}}{G_2^3}$$

vel:

$$x^m y^{n-1} (n\, G(xy)_2 - y\, G(xy)_{2\,2}) = (\phi + \psi_1)\, G_2^2 - G_2\, (\psi\, G_{2\,1} + \psi_2\, G_1) + \psi\, G_1\, G_{2\,2}$$

vel etiam

$$\frac{-G_{2\,2}\,(\psi\, G_1 + x^m y^n)}{G_2} = (\phi + \psi_1)\, G_2 - (\psi\, G_{2,1} + \psi\, G_1) - n\, x^m y^{n-1}$$

In omnibus locis, ubi G_2 evanescit, non evanescit $G_{2,2}$. Si enim hoc fieret, haberemus aeqationem conditionalem inter coëfficientes aequationis nostrae $G(x\,y) = O$, quos plane generales esse supposuimus. Pro iisdem valoribus non fit infinitum membrum aequationis nostrae ad dextram situm, unde concluditur, necesse esse, ut sit:

$$\psi\, G_1 + x^m y^n = O \text{ ubi } G_2 = O,$$

ne aequatio turbetur.

Sint χ et ϑ functiones quaedam integrae quantitatum x et y; habebimus aequationem identicam:

15) $\qquad \chi\, G + \vartheta\, G_2 + \psi\, G_1 + x^m y^n = O$

Reperimus, functionem ψ hanc exhibere formam:

$$\psi\,(xy) = (p_1\, x + p_0)\,(x\,y^2 + dy) + (q_1\, x + q_2\, x^2 + q_3\, x^3)\, y + r_0 + r_1\, x + r_2\, x^2 + r_3\, x^3$$

Quum autem sit $p_1 = O$, (quod statim demonstrabitur), licet ψ tali modo scribere:

$$\psi\,(x\,y) = p\, x\,y^2 + (q_3\, x^3 + q_2\, x^2 + q_1\, x + p\,d)\, y + r_3\, x^3 + r_2\, x^2 + r_1\, x + r_0$$

continet autem functio ψ octo coëfficientes determinandos. Ponatur nunc, si signum $(\nu$ functionem integram variabilis x, ν^{ti} gradus designat:

$$\chi = y^2\,(o) + y\,(2) + (2)$$
$$\vartheta = y^2\,(2) + y\,(2) + (3)$$

Continet functio χ 7, ϑ autem 10 incognitos coëfficientes, habemus igitur $8 + 7 + 10 = 25$ quantitates definiendas. Substitutis valoribus functionum ψ, χ, ϑ in 14), quum sit

$$G(xy) = xy^3 + y^2\,(ax + d) + y\,(x^2 + bx + g) + x^3 + cx + k = x\,y^3\,(0) + y^2\,(1) + y\,(2) + (3)$$
$$G(xy)_1 = y^3 + ay^2 + y\,(2x + b) + 3x^2 + c = \;.\;.\;.\;.\;.\; = y^3\,(0) + y^2\,(0) + y\,(1) + (2)$$
$$G(xy)_2 = 3y^2\,x + 2y\,(ax + d) + x^2 + bx + g = \;.\;.\;.\;.\;.\; = xy^2\,(0) + y\,(1) + (2)$$
$$\psi\,(xy) = xy^2\,(0) + y\,(3) + (3)$$

accipiemus:

$$y^5\, x(0) + y^4\,(3) + y^3\,(3) + y^2\,(4) + y\,(5) + (5) = -x^m y^n \qquad \left(\begin{matrix} m \\ n \end{matrix}\right\} = 0,\,1,\,2,\,3\,\Big)$$

Aequatio haec continet $1 + 2.\ 4 + 5 + 2.\ 6 = 26$ coëfficientes, quorum unus jam statim identice evanescit. Quum aequatio ipsa identica esse debeat, comparatis reliquis 25 coëfficientibus utriusque partis inter se, fient 25 aequationes lineares, quibus nostri 25 incogniti coëfficientes plane definiuntur. Facillime enim intelligitur, a e q u a t i o n u m n o s t r a r u m s y s t e m a t i s d e t e r m i n a n t e m n o n e v a n e s c e r e; est enim ipsa „resultans eliminationis“ quae vocatur, aequationum $G = 0$, $G_1 = 0$, $G_2 = 0$. Donec igitur non

est $G_1=0$ una cum $G_2=0$ id est, donec coëfficientes aequationis $G(xy)=0$ plane sunt generales, valent omnia, quae demonstramus. Si determinans illa evanescit, afficitur curva, quae per $G(xy)=0$ repraesentatur, singularitatibus et ordo (Rang) ipsius diminuitur.

Scribamus brevitatis causa:

$$G(xy)=y^3c_0+y^2c_1+yc_2+c_3 \qquad \psi(xy)=\psi_2y^2+\psi_1y+\psi_0$$
$$G(xy)_1=y^3+ay^2+yc'_2+c'_3 \qquad \chi(xy)=\chi_2y^2+\chi_1y+\chi_0$$
$$G(xy)_2=3y^2c_0+2yc_1+c_2 \qquad \vartheta(xy)=\vartheta_2y^2+\vartheta_1y+\vartheta_0$$

erit observandum

$$(\chi_2y^2+\chi_1y+\chi_0)(y^3c_0+y^2c_1+yc_2+c_3)+(\psi_2y^2+\psi_1y+\psi_0)(y^3+y^2c'_1+yc'_2+c'_3)$$
$$+(\vartheta_2y^2+\vartheta_1y+\vartheta_0)(3y^2c_0+2yc_1+c_2)=-x^my^n$$

vel:

$$y^5(\psi_2+\chi_2c_0)+y^4(a\psi_2+\psi_1+\chi_2c_1+\chi_1c_0+3\vartheta_2c_0)$$
$$+y^3(\psi_2c'_2+a\psi_1+\psi_0+\chi_2c_2+\chi_1c_1+\chi_0c_0+2\vartheta_2c_1+3\vartheta_1c_0)$$
$$+y^2(\psi_2c'_3+\psi_1c'_2+a\psi_0+\chi_2c_3+\chi_1c_2+\chi_0c_1+\vartheta_2c_2+2\vartheta_1c_1+3\vartheta_0c_0)$$
$$+y(\psi_1c'_3+\psi_0c'_2+\chi^1c_3+\chi_0c_2+\vartheta_2c_2+2\vartheta_0c_1)$$
$$+\psi_0c'_3+\chi_0c_3+\vartheta_0c_2+x^my^n=O$$

Unusquisque coëfficiens potestatum y evanescit identice. Si in aequationibus, quas hoc modo accipimus, pro c, ψ, χ, ϑ valores eorum

$$c_0=x$$
$$c_1=ax+d \qquad \psi_2=px \qquad \chi_2=a \qquad \vartheta_2=\varepsilon_0+\varepsilon_1x+\varepsilon_2x^2$$
$$c_2=x^2+bx+g,\ \psi_1=pd+q_1x+q_2x^2+q_3x^3;\ \chi_1=\beta_0+\beta_1x+\beta_2x^2,\ \vartheta_1=\zeta_0+\zeta_1x+\zeta_2x^2$$
$$c_3=x^3+cx+k,\ \psi_0=r_0+r_1x+r_2x^2+r_3x^3,\ \chi_3=\gamma_0+\gamma_1x+\gamma_2x^2,\ \vartheta_0=\eta_0+\eta_1x+\eta_2x^2+\eta_3x^3$$

substituantur, jam iterum omnis coëfficiens omnium potestatum x per se nullitati aequandus erit. Jam habebimus:

Ia) (Coeff. potest y^5x) $p+\alpha=0$

I) 1) „ „ y^4x^3 $\beta_2+3\varepsilon_2+q_3=0$

2) „ „ „ x^2 $\beta_1+3\varepsilon_1+q_2=0$

3) „ „ „ x $\beta_0+3\varepsilon_0+q_1=0$

4) „ „ „ 1 $-pd+pd=0$ evanescit identice!

II) 1) „ „ y^3x^3, $a\beta_2+\gamma_2+2a\varepsilon_2+3\zeta_2+aq_3+r_3=(m,n)_{3,3}$

2) „ „ „ x^2 $-p+a\beta_1+d\beta_2+\gamma_1+2a\varepsilon_1+2d\varepsilon_2+3\zeta_1+2p+aq_2+r_5=(m,n)_{2,3}$

3) „ „ „ x $-pb+a\beta_0+d\beta_1+\gamma_0+2a\varepsilon_0+2d\varepsilon_1+3\zeta_0+bp+aq_1+r_2=(m,n)_{1,3}$

4) „ „ „ 1 $-pg+d\beta_0+2d\varepsilon_0+apd+r_0=(m,n)_{0,3}$

$$(m,n)_{m,n}=-1$$
$$(m,n)_{m_1,n_1}=0 \text{ si } m_1 \gtrless m,\ n_1 \lessgtr n$$

III. 1) Coeff. potest. y^2x^4 $\beta_2+\varepsilon_2+3\eta_3+2q_3=O$

2) „ „ „ x^3 $b\beta_2+\beta_1+a\gamma_2+b\varepsilon_2+\varepsilon_1+2a\zeta_2+3\eta_2+2p+bq_3+2q_2+ar_3=(m,n)_{3,2}$

3) „ „ „ x^2 $g\beta_2+b\beta_1+\beta_0+d\gamma_2+a\gamma_1+g\varepsilon_2+b\varepsilon_1+\varepsilon_0+2d\zeta_2+2a\zeta_1+3\eta_1+bq_2$
$+2q_1+ar_2=(m,n)_{2,2}$

4) „ „ „ x $g\beta_1+b\beta_0+d\gamma_1+a\gamma_0+g\varepsilon_1+b\varepsilon_0+2a\zeta_0+2d\zeta_1+3\eta_0+bq_1+2pd$
$+ar_1=(m,n)_{1,2}$

5) „ „ „ 1 $g\beta_0+d\gamma_0+g\varepsilon_0+2d\zeta_0+(bd-k)p+ar_0=(m,n)_{0,2}$

IV. 1) „ „ yx^5 $\beta_2+3q_3=O$

2) „ „ „ x^4 $\beta_1+\gamma_2+\zeta_2+2a\eta_3+3q_2+2r_3=O$

3) „ „ „ x^3 $\beta_0+c\beta_2+\gamma_1+b\gamma_2+\zeta_1+b\zeta_2+2a\eta_2+2d\eta_3+3q_1+cq_3+2r_2+br_3$
$=(m,n)_{3,1}$

4) „ „ „ x^2 $c\beta_1+k\beta_2+\gamma_0+b\gamma_1+\gamma_2+\zeta_0+b\zeta_1+g\zeta_2+2a\eta_1+2d\eta_2+3q_1+cq_2$
$+2r_1+br_2=(m,n)_{2,1}$

5) „ „ „ x $c\beta_0+k\beta_1+b\gamma_0+g\gamma_1+b\zeta_0+g\zeta_1+2a\eta_0+2d\eta_1+cq_1+2r_0+br_1$
$=(m,n)_{1,1}$

6) „ „ „ 1 $k\beta_0+g\gamma_0+g\zeta_0+2d\eta_0+cdp+br_0=(m,n)_{0,1}$

V. 1) „ „ x^5 $\gamma_2+\eta_3+3r_3=O$

2) „ „ x^4 $\gamma_1+\eta_2+b\eta_3+3r_2=O$

3) „ „ x^3 $\gamma_0+c\gamma_2+\eta_1+b\eta_2+g\eta_3+3r_1+cr_3=(m,n)_{3,0})$

4) „ „ x^2 $c\gamma_1+k\gamma_2+\eta_0+b\eta_1+g\eta_2+3r_0+cr_2=(m,n)_{2,0}$

5) „ „ x $c\gamma_0+k\gamma_1+b\eta_0+g\eta_1+cr_1=(m,n)_{1,0}$

6) „ „ 1 $k\gamma_0+g\eta_1+cr_0=(m,n)_{0,0}$

Eliminatis quantitatibus $\alpha, \beta, \gamma, \varepsilon, \zeta, \eta$ aequationes formemus, quae quantitates p, q_1, r solas contineant. Reperimus successive: *)

Cf.: Ia. $\alpha=-p$ II, 1 $\zeta_2=\frac{6a+1}{27}q_3+\frac{2}{3}r_3+\frac{1}{3}A$

IV. 1. $\beta_2=-3q_3$ IV, 2 $\beta_1=\frac{2-12a}{27}q_3-3q_2+\frac{1}{3}r_3-\frac{1}{3}A$

I. 1. $\varepsilon_2=\frac{2}{3}q_3$ I. 2. $\varepsilon_1=\frac{12a-2}{81}q_3+\frac{2}{3}q_2-\frac{1}{9}r_3+\frac{1}{9}A$

III. 1. $\eta_3=\frac{1}{9}q_3$ I,3 $\left|\beta_0=3\frac{g-ad}{d}p+2q_1-\frac{3r_0}{d}+\frac{3D}{d}\right.$

II, 4 $\left|\varepsilon_0=-q_1+\frac{ad-g}{d}p+\frac{r_0}{d}-\frac{D}{d}\right.$

III, 2 $\eta_2=-\frac{2}{3}p+\frac{-36a^2+27a+108b-4}{243}q_3+\frac{1}{9}q_2+\frac{6a-2}{27}r_3+\frac{2-6a}{27}A+\frac{1}{3}E,$

V. 2 $\gamma_1=\frac{2}{3}p+\frac{36a^2-27a-135b+4}{243}q_3-\frac{1}{9}q_2+\frac{2-6a}{27}r_3-3r_2+\frac{6a-2}{27}A-\frac{1}{3}E$

---

*) Brevitatis causa scribemus: *A, B, C, D*; *E, F, G, H*; *K, L, M, N*; *P, Q, R, S* pro $(m,n)_{m,n}$
cum indice resp.: 3,3 2,3 1,3 0,3 3,2 2,2 1,2 0,2 3,1 2,1 1,1 0,1 3,0 2,0 1,0 0,0

II, 2 $$\zeta_1 = -\frac{5}{9} + p\,\frac{21a + 135b + 405\,d - 4}{729}\,q_3 + \frac{6a+1}{27}\,q_2 + \frac{3a-2}{81}\,r_3 + \frac{2}{3}\,r_2 + \frac{2-3a}{81}\,A + \frac{B}{3} + \frac{1}{9}\,E,$$

II, 3; III, 5 $$\gamma_0 = \frac{8adg - 2a^2d^2 - 3bd^2 + 6g^2 + 3dk}{d^2}\,p + \frac{4d - 24ad}{81}\,q_3 - \frac{10d}{3}\,q_2 + \frac{2ad - 3g}{d}\,q_1 + \frac{2d}{9}\,r_3 + 2r_1 + \frac{6g - 5ad}{d^2}\,r_0 - \frac{2d}{9}\,A - 2C + \frac{2ad - 6g}{d^2}\,D + \frac{3}{d}\,H,$$

II, 3; III, 5 $$\zeta_0 = \frac{a^2d^2 - 3adg + 2g^2 + bd^2 + 3kd}{d^2}\,p + \frac{12a-2}{81}\,d\,q_3 + \frac{5}{3}\,dq_2 + \frac{g - ad}{d}\,q_1 - \frac{d}{9}\,r_3 - r_1 + \frac{2ad - 2g}{d^2}\,r_0 + \frac{d}{9}A + C + \frac{2g - ad}{d^2}\,D - \frac{1}{d}\,H$$

V, 3 $$\eta_1 = \frac{1}{3d^2}\left(6a^2d^2 - 24adg + 11bd^2 - 18g^2 - 9dk\right)p + \frac{1}{243}\left(36a^2b - 27ab + 72ad - 108b^2 + 4b + 27c - 12d + 27g\right)q_3 + \frac{30d - b}{9}\,q_2 + \frac{3g - 2ad}{d}\,q_1 + \frac{-6ab + 2b + 54c - 6d}{27}\,r_3 - 5r_1 + \frac{5ad - 6g}{d^2}\,r_0 + \frac{6ab - 2b + 6d}{27}\,A + \frac{6 - b}{3}\,C + \frac{6g - 2ad}{d^2}\,D - \frac{3}{d}\,H + P$$

V, 6 $$\eta_0 = \frac{2a^2d^2k - 8adgk + 3bd^2k - 6g^2k - 3dk^2}{gd^2}\,p + \frac{-4dk + 24adk}{81g}\,q_3 + \frac{10dk}{3g}\,q_2 + \frac{-2adk + 3gk}{gd}\,q_1 - \frac{2dk}{9g}\,r_3 - \frac{2kr_1}{g} + \frac{-2adk + 6gk}{gd^2}\,D + \frac{2dk}{9g}\,A + \frac{2kC}{g} - \frac{3k}{dg}\,H + \frac{S}{g} + \frac{-6gk + 5adk - cd^2}{d^2y}\,r_0$$

Substitutis his valoribus in reliquas aequationes nostras accipiemus:
ex aequatione III, 3

$$\frac{1}{9d^2}\left(-216adg + 54a^2d^2 + 99bd - 162g^2 - 81dk - 22ad^2 + 18dg\right)p + \frac{q_3}{729}\left\{108a^3 - 39a^2 + 324a^2b - 594ab + 1782ad - 972b^2 + 4a)\right\} + 72b + 243c - 135d - 1944g$$

$$+ \frac{q_2}{27}\left\{12a^2 - a - 45b + 270d\right\} + \frac{q_1}{d}\,(9g - 6ad + 3d) + \frac{r_3}{81}\,(-12a^2 - 54ab + 2a + 36b + 486c - 189d) + \frac{10a}{3}\,r_2 - 15r_1$$

$$+\frac{1}{d^2}(15ad-18g-2d)+\frac{A}{81}(12a^2+54ab-2a-36b+108d)$$

$$+\frac{2aB}{7}+(6-b)C+\frac{-6ad+18g+2d}{d^2}D$$

$$-\frac{a}{9}E-\frac{9}{d}H-F+3P=0$$

III, 4

$$\frac{p}{9d^2g}\left\{54a^2d^2k+18a^2dg^2-27abd^2g-207adgk+90ag^3+81bd^2k+18bdg^2+14d^3g-81dk^2-162g^2k\right\}$$

$$+\frac{q_3}{729g}\left\{108a^2dg-39adg+648adk-216ag^2-135bdg+810d^2g+4dg-108dk+36g^2\right\}$$

$$+\frac{q_2}{27g}\left\{12adg+270dk-dg-63g^2\right\}$$

$$+\frac{q_1}{gd}\left\{-6adk-ag^2+2bdg+9gk\right\}+\frac{r_3}{81g}\left\{-12adg-2dg-54dk-18g^2\right\}-\frac{5d}{3}r_2+\frac{ag-6k}{g}r_1$$

$$+\frac{r_0}{d^2g}\left\{-a^2dg+15adk+2ag^2-2bdg-3cd^2-18gk\right\}+\frac{A}{81}\left\{12adg-2dg+54dk-18g_2\right\}$$

$$+\frac{2d}{3}B+\frac{6ck}{g}C+\frac{-6adk-2ag^2+2bdg+18gk}{gd^2}D$$

$$+\frac{5d}{9}E+\frac{ag-9k}{gd}H+\frac{3S}{g}=0$$

IV, 3

$$\frac{1}{9d}\left\{-39ad+d+27g\right\}p+\frac{q_3}{729}\left\{-216a^3+270a^2-486ab-36a-324b-1458c+567d+8\right\}$$

$$+\frac{12a-2}{27}q_2+5q_1$$

$$+\frac{r_3}{81}\left\{36a^2-27a-108b+4\right\}-\frac{1}{3}r_2+\frac{3r_0}{d}+\frac{A}{81}\left\{-36a^2+27a+27b-4\right\}$$

$$+\frac{B}{3}+\frac{3D}{d}+\frac{6a-2}{9}E-K=0$$

IV, 4

$$\frac{p}{9d^2}\left\{-36a^3d^2-9a^2d^2-144a^2dg-66abd^2+45adg-54adk-108ag^2-17bd^2+72g^2-12d^3+18dk\right\}$$

$$+\frac{q_3}{729}\left\{216a^3b-270a^2b+216a^2d-648ab^2-36ab-162ac-18ad-270b^2+1053bd+8b\right.$$

$$\left.+54c-6d-54g-2187k\right\}$$

$$+\frac{q_2}{27}\left\{180ad-2b-39d-54c\right\}+\frac{q_1}{d}\left\{-4a^2d+ad+6ag-2g+3d\right\}$$

$$+\frac{r_3}{81}\left\{-36a^2b-3ab-324ac+4b-189g+27c+3d\right\}$$

$$-\frac{4b}{3}r_2+(3-10a)r_1+\frac{1}{d^2}(10a^2d-6ag-3ad+4g)r_0+\frac{1}{81}\left\{+36a^2b+3ab-4b+3d-27c+27g\right\}$$

$$+\frac{b}{3}B+\frac{12a-2ab-3}{3}C$$

$$+\frac{1}{d^2}(-3ad+12ag-4g)D=0$$

IV, 5

$$\frac{p}{9\,g\,d^2}\Big\{36\,a^3 d^2 k - 9\,a^2 b\,d^2 g - 144\,a^2 d\,g\,k - 36\,a^2 d^3 g + 54\,a\,b\,d^2 k + 45\,ab\,dg^2 - 27\,ac\,d^2 g - 144\,a\,d^2 g^2 - 108\,a\,g^2 k - 54\,a\,k^2 d - 18\,b^2 d^2 g + 66\,b\,d^3 g + 18\,bdgk + 72\,b\,g^3 - 54\,d^2 g\,k - 108\,d\,g^3 + g^2 d^2\Big\}$$

$$+ \frac{q_3}{729\,g}\Big\{216\,a^2 bdg + 432\,a^2 d\,k + 108\,a^2 g^2 - 270\,abdg + 432\,a\,d^2 g - 72\,adk - 60\,ag^2 - 324\,agk - 648\,b^2 dg + 42\,bdg - 270\,bg^2 + 162\,cdg - 72\,d^2 g + 243\,dg^2 + 8\,g^2 + 54\,gk\Big\}$$

$$+ \frac{q_2}{27\,g}\Big\{6\,ag^2 + 180\,adgk - 51\,b\,d\,g + 180\,d^2 g - g^2 - 81\,gk\Big\}$$

$$+ \frac{q_1}{gd}\Big\{-4\,a^2 dk + ab\,dg - 4\,ad^2 g + 6\,agk - 2\,bg^2 + 3\,cdg + 6\,g^2 d\Big\}$$

$$+ \frac{r_3}{81\,g}\Big\{-36abdg - 36adk + 15ag^2 + 21bdg + 324cdg - 36d^2 g + 4g^2 + 27gk\Big\} - \frac{7\,g}{3}\,r_2 - \frac{4\,ak + 10\,dg}{g}\,r_1$$

$$+ \frac{r_0}{d^2 g}\Big\{10\,a^2 d\,k - 10\,a\,d^2 g - 3ab\,dg - 2\,ac\,d^2 - 12\,agk + 4\,b\,g^2 - 3\,c\,d\,g - 12\,d\,g^2 + 2\,d^2 g\Big\}$$

$$+ \frac{A}{81\,g}\Big\{15\,ag^2 + 36\,abdg - 36\,adk - 21bdg + 36\,d^2 g - 27\,g\,k - 4g^2\Big\}$$

$$+ \frac{g}{3}\,B + \frac{C}{6\,g}\Big\{-24\,ak - 6\,b\,g - 2bdg + 12\,dg\Big\} + \frac{D}{d^2}\Big\{-4\,ad^2 + abd - 4\,bg + 3\,cd + 12\,dg\Big\} - \frac{2\,g}{9}\,E$$

$$+ \frac{H}{dg}\,(-6ak + 2\,bg - 6dg) - M + 2\,dP - \frac{2\,a\,S}{g} = 0$$

IV, 6

$$\frac{p}{gd^2}\Big\{4\,a^2 d^3 k - a^2 d^2 g^2 - 16\,ad^2 gk + 5\,a\,d\,g^3 - 3\,a\,d^2 g\,k + 6\,b\,d^3 k - 12\,d\,g^2 k - 6\,d^2 k^2 - 2\,b\,d^2 g + 5\,d\,g\,k + 8\,g^3\Big\}$$

$$+ \frac{q_3}{81\,g}\Big\{-12adg^2 + 48ad^2 gk - 8d^2 k - 2dg^2\Big\} + \frac{q_2}{3g}\Big\{20\,d^2 k + 15\,d\,g^2\Big\} + \frac{-4ad^2 k + adg^2 + 8dgk - 2g^3}{g\,d}\,q_1$$

$$+ \frac{dg^2 - 4\,d^2 k}{9\,g}\,r_3 + \frac{g^2 - 4\,d\,k}{g}\,r_1$$

$$+ \frac{r_0}{gd^2}\Big\{10\,a\,d^2 - 3\,adg^2 + bg\,d^2 - 2\,cd^3 + 15\,dgk + 4\,g^3\Big\} + \frac{4\,d^2 k - g\,d}{9\,g}\,A + \frac{4\,dk - g^2}{g}\,C$$

$$+ \frac{-4\,ad^2 k + a\,d\,g^2 + 15\,dgk - 4\,g^3}{gd^2}\,D$$

$$+ \frac{2\,g^2 - 6\,d\,k}{d\,g}\,H + \frac{2\,d\,S}{g} - N = 0$$

V, 4

$$\frac{p}{3gd^2}\Big\{6\,a^2 b\,d^2 g + 6\,a^2 d^2 k - 24\,ab\,dg^2 - 24\,adgk + 11\,b^2 d^2 g + 9\,bd^2 k - 18\,bg^3 - 9\,bd\,gk + 2\,cd^2 g - 2\,d^2 g^2 - 18\,g^2 k - 9\,d\,k^2\Big\}$$

$$+ \frac{q_3}{243\,g}\Big\{36\,a^2 b^2 g + 36\,a^2 cg - 36\,a^2 g^2 - 27\,a\,b^2 g - 27\,acg + 72\,adk + 216\,ab\,dg + 27\,ag^2 - 108\,b^3 g + 4b^2 g - 12\,bdg - 108\,bcg + 81\,b\,g^2 + 4\,c\,g - 12\,d\,k - 4\,g^2 - 27\,g\,k\Big\}$$

$$+ \frac{q_2}{9g}\Big\{30\,b\,d\,g - b^2 g + 30\,dk - cg + g^2\Big\} + \frac{q_1}{gd}\Big\{-2\,ab\,dg - 2\,a\,d\,k + 3\,bg^2 + 3gk\Big\}$$

$$+ \frac{r_3}{27\,g}\Big\{6\,a\,b^2 g - 6\,acg + 6\,a\,g^2 + 2\,b^2 g + 54\,bcg - 6b\,dg + 2\,cg - 2\,g^2 - 6dk - 81gk\Big\}$$

$$-2cr_2 - \frac{5bg+2k}{g} r_1 + \frac{r_0}{d^2 g}\left\{5abdg + 5adk - 6bg^2 - cd^2 - 6gk + 3gd^2\right\}$$
$$+ \frac{A}{27}\left\{6ab^2 + 6ac - 6ag - 2b^3 + 6bd + 6dk - 2c + 2g\right\}$$

$$+ \frac{-b^2g + 6bg + 6k}{3g} C + \frac{1}{gd^2}\left\{-2abdg - 2adk + 6bg^2 + 6d^2gk\right\} D$$
$$+ \frac{g-c}{3} E - \frac{3bg+3k}{dg} H + P + \frac{S}{g} - Q = 0$$

V, 5

$$\frac{p}{3gd^2}\left\{6a^2bd^2k - 6a^2cd^2g + 6a^2d^2g^2 - 24abdgk + 24acdg^2 - 24adg^3 + 9b^2d^2k - 9bcd^2g\right.$$
$$\left. + 11bd^2g - 18bg^2k - 9bk^2d + 18cg^3 + 9cdgk + 2d^2gk + 9dg^2k - 18g^4\right\}$$

$$+ \frac{q_3}{243g}\left\{36a^2bg^2 + 36a^2gk + 72abdk - 27abg^2 - 27acdg + 72adg^2 - 27agk - 108b^2g^2 - 12bdk\right.$$
$$\left. + 4by^2 - 135bgk + 12cdy + 27cg^2 - 12dg^2 - 27g^3 + 4gk\right\}$$

$$+ \frac{q_2}{9g}\left\{30bdk + 30dg^2 - bg^2 - 30cdy - gk\right\} + \frac{q_1}{gd}\left\{-2abdk + 2acdg - 2adg^2 + 3bgk - 3cg^2 + 3g^3\right\}$$

$$+ \frac{r_3}{27g}\left\{-6abg^2 - 6agk - 6bdk + 2bg^2 + 6cdg + 54cg^2 - 6dg^2 + 2gk\right\} - 3kr_2 + \frac{r_1}{g}\left\{-2bk + 3cg - 5g^2\right\}$$

$$+ \frac{r_0}{gd^2}\left\{5abdgk + 5adg^2 - 5acdg - bcd^2 - 6bgk + 6cg^2 - 6g^3\right\}$$
$$+ \frac{A}{27g}\left\{6abg^2 + 6agk + 6bdk - 2bg^2 - 6cdg + 6dg^2 - 2gk\right\}$$

$$+ \frac{C}{3g}\left\{-bg^2 + 6bk - 6cg + 6g^2\right\} + \frac{D}{gd^2}\left\{-2abdk - 2adg^2 - 2acdg + 6bgk - 6cg^2 + 6g^3\right\}$$
$$- \frac{k}{3} E + \frac{3c-3g}{d} H + Pg - R + \frac{b}{g} S = 0$$

Aequationes hae suppeditant nobis in omni casu speciali valores coëfficientium functionis $\psi(xy)_{m,n}$. Fac datos esse hos valores; jam coefficientes $l^{(\alpha)}_{m,n}$ lineariter per illos exprimes, si aequationis nostrae:

$$F(x_t y_t)_{m,n} \frac{dx_t}{dt} = \frac{\phi(x_t y_t)_{m,n}}{G(x_t y_t)_2} \frac{dx_t}{dt} + \frac{d}{dt} \frac{\psi(x_t y_t)_{m,n}}{G(x_t y_t)_2}$$

utramque partem pro infinitis valoribus x, y in seriem evolveris. Est autem:

$$\frac{\psi(\bar{x}_t \bar{y}_t)}{G(\bar{x}_t \bar{y}_t)_2} = \frac{(p_1 \bar{x}_t^2 + p_0 \bar{x}_t)\bar{y}_t^2 + (q_3 \bar{x}_t^3 + q_2 \bar{x}_t^2 + (q_1 + p_1 d)\bar{x}_t + p_0 d)\bar{y}_t + r_3 \bar{x}_t^3 + r_2 \bar{x}_t^2 + \ldots}{G(\bar{x}_t \bar{y}_t)_2}$$

$= p_1$	$\frac{1}{3} t^{-3} - \frac{1}{9} t^{-2} + \frac{6a-1}{27} t^{-1} + \frac{54a^2-36a-81b+5}{729} t + \text{const. } t^2 + \text{const. } t^4 + \dots$
p_0	$\frac{1}{3} - \frac{1}{9} t + \frac{6a-1}{27} t^2 + \frac{54a^2-36a-81b+5}{729} t^4 + \dots$
q_3	$-\frac{1}{3} t^{-4} + \frac{1-3a}{27} t^{-2} + \frac{4-18a}{243} t^{-1} + \left. \begin{matrix} \frac{-108a^2+60a+162b}{2187} \\ \frac{-243d-8}{2187} \end{matrix} \right\vert t + \frac{405a^3-810a^2-\dots-35}{19683} t^{-2} + \frac{1701a^4+\dots77}{3.\,59049} t^4 + \dots$
q_2	$-\frac{1}{3} t^{-1} + \frac{1-3a}{27} t + \frac{4-18a}{243} t^2 + \left. \begin{matrix} \frac{-108a^2+60a+162b}{729} \\ \frac{-243d-8}{729} \end{matrix} \right\vert t^4 + \dots$
$q_1 + p_1 d$	$-\frac{1}{3} t^2 + \frac{1-3a}{27} t^4 + \dots$
$p_0 d$	$t^5 (\dots)$
r_3	$\frac{1}{3} t^{-2} + \frac{1}{9} t^{-1} + \frac{18a-5}{243} t + \frac{-27a^2+36a+81b-7}{729} t^{-2} + \frac{-324a^3+810a^2 + \dots + 44}{19683} t^4 + \dots$
r_2	$+\frac{1}{3} t + \frac{1}{9} t^2 + \frac{18a-5}{243} t^4 + \dots$
r_1	$+\frac{1}{3} t^4 + \dots$
r_0	$+ t^7$

Tum erit

$$\frac{d}{dt} \frac{\psi\,(\overline{x_t}\ \overline{y_t})}{G\,(x_t\ y^t)} = \frac{4}{3} q_3 t^{-5} - p_1 t^{-4} + \left\{ \frac{2}{9} p_1 + \frac{6a-2}{27} q_3 - \frac{2}{3} r_3 \right\} t^{-3}$$

$$+ \left\{ \frac{1-6a}{27} p_1 + \frac{18a-4}{243} q_3 + \frac{1}{3} q_2 - \frac{1}{9} r_3 \right\} t^{-2}$$

$$+ \left\{ \frac{54a^2+\dots+5}{729} p_1 - \frac{1}{9} p_0 + \frac{-108a^2+60a+162b-243d-8}{2187} q_3 + \frac{1-3a}{27} q_2 + \frac{18a-5}{243} r_3 + \frac{1}{3} r_2 \right\}$$

$$+ \left\{ \text{const.} p_1 + \frac{12a-2}{27} p_0 + 2\,\frac{405a^3-810a^2+\dots-35}{19683} q_3 + \frac{8-36a}{243} q_2 - \frac{2}{3}(q_1+p_1 d) + \frac{-54a^2+72a+162b-14}{729} r_3 + \frac{2}{9} r_2 \right\} t$$

$$+ \left\{ \text{Const.} p_1 + \frac{216a^2-144a-324b+20}{729} p_0 + 4\,\frac{1701a^4+\dots+77}{3.\,59049} q_3 + \frac{-432a^2+\dots-32}{729} q_2 + \frac{4-12a}{27}(q_1+p_1 d) + 4\,\frac{-324a^3+\dots+44}{19643} r_3 + \frac{72a-20}{243} r_2 + \frac{4}{3} r_1 \right\} t^3$$

$$+ t^4\, \mathfrak{P}(t)$$

Simili modo erit

$$\frac{\phi\,(\overline{x_t}\ \overline{y_t})}{G\,(\overline{x_t}\ \overline{y_t})_2} \frac{d\,\overline{x_t}}{dt} = \frac{-3\,t^{-4}}{G\,(\overline{x_t}\ \overline{y_t})_2} \left\{ l^{(6)} x^2 y + l^{(5)} x^3 + l^{(4)} xy + l^{(3)} x + l^{(2)} y + l^{(1)} \right\}$$

$$
= \begin{array}{l|llllll}
l^{(6)} & t^{-5} & +\frac{3a-1}{9}t^{-3} & +\frac{18-4}{81}t^{-2} & +\frac{108a^2-60a-162b+243d+8}{729} & -\frac{405a^3+..+35}{6561}t & -\frac{1701a^4-..+77}{59049}t^3 + .. \\
l^{(5)} & & -t^{-3} & -\frac{1}{3}t^{-2} & +\frac{5-18a}{81} & +\frac{27a^2-36a-81b+7}{243}t & -\frac{-324a^3+..+44}{6561}t^3 + .. \\
l^{(4)} & & & +t^{-2} & +\frac{3a-1}{9} & +\frac{18a-4}{81}t & +\frac{108a^2-60a..+8}{729}t^3 + .. \\
l^{(3)} & & & & -1 & -\frac{1}{3}t & +\frac{5-18a}{81}t^3 \quad + .. \\
l^{(2)} & & & & & +t & +\frac{3a-1}{9}t^3 \quad + .. \\
l^{(1)} & & & & & & -t^3 \quad + .. \\
\hline
\end{array}
$$

$$
= l^{(6)}t^{-5} + \left\{\frac{3a-1}{9}l^{(6)} - l^{(5)}\right\}t^{-3} + \left\{\frac{18a-4}{81}l^{(6)} - \frac{1}{3}l^{(5)} + l^{(4)}\right\}t^{-2}
$$

$$
+ \left\{\frac{108a^2-60a-162b+243d+8}{729}l^{(6)} + \frac{5-18a}{81}l^{(5,} + \frac{3a-1}{9}l^{(4)} - l^{(3)}\right\}
$$

$$
+ \left\{-\frac{405a^3-..-35}{6561}l^{(6)} + \frac{27a^2-36a-81b+7}{243}l^{(5)} + \frac{18a-4}{81}l^{(4)} - \frac{1}{3}l^{(3)} + l^{(2}\right\}t
$$

$$
+ \left\{-\frac{1701a^4-..-77}{59049}l^{6} - \frac{-324a^3+810a^2+..+44}{6561}l^{5} + \frac{108a^2-60a-162b+243d+8}{729}l^{4)} + \frac{5-18a}{81}l^{3} + \frac{3a-1}{9}l^{(2)} - l^{(1)}\right\}t^3
$$

$$
+ \ldots.
$$

Alia ex parte posuimus:

$$
F(\overline{x}_t\,\overline{y}_t)_{m,n}\frac{d\overline{x}_t}{dt} = (m,n)_{-3}\,t^{-3} + (m,n)_{-2}\,t^{-2} + (m,n)_0 + (m,n)_1\,t + \ldots
$$

Comparatis coëfficientibus earundem potestatum variabilis t erit

$$
l^{(6)} + \frac{4}{3}q_3 = 0
$$

$$
-p_1 = 0
$$

Propter hanc aequationem omittemus in iis quae sequuntur omnia membra in p_1 multiplicata, pro p_0 autem p scribemus.

$$
\frac{3a-1}{9}l^{(6)} - l^{(5)} + \frac{6a-2}{27}q_3 - \frac{2}{3}r_3 = (m,n)_{-3}
$$

$$
\frac{18a-4}{81}l^{(6)} - \frac{1}{3}l^{(5)} + l^{(4)} + \frac{18a-4}{243}q_3 + \frac{1}{3}q_2 - \frac{1}{9}r_3 = (m,n)_{-2}
$$

$$
\frac{108a^2-60a-162b+243d+8}{729}l^{(6)} + \frac{5-18a}{81}l^{(5)} + \frac{3a-1}{9}l^{(4)} - l^{(3)} - \frac{1}{9}p
$$

$$
+ \frac{-108a^2+60a+162b-243d-8}{2187}q_3 + \frac{1-3a}{27}q_2 + \frac{18a-5}{243}r_3 + \frac{r_2}{3} = (m,n)_0
$$

$$\left.\begin{aligned}
&\frac{1}{6561}\left\{-405a^3+810a^2+1458ab-315a-729b-2187c+1458d+35\right\}l^{(6)}\\
&\qquad+\frac{1}{243}\left\{27a^2-36a-81b+7\right\}l^{(5)}+\frac{18a-4}{81}l^{(4)}-\frac{1}{3}l^{(3)}+l^{(2)}\\
&+\frac{12a-2}{27}p+\frac{1}{19683}\left\{810a^3-1620a^2-2916ab+630a+1458b+4374c-2916d-70\right\}q_3\\
&\qquad+\frac{8-36a}{81}q_2-\frac{2}{3}q_1+\frac{-54a^2+72a+162b-14}{243}r_3-\frac{2}{9}r_2
\end{aligned}\right\}=(m,n)_1$$

$$\left.\begin{aligned}
&\frac{1}{59049}\left\{\begin{aligned}&-1701a^4+5676a^3+8748a^2b-3780a^2-11664ab-6561b^2-13122ac+17496ad+924a\\&+2268b+10935c-4860d-13122g-77\end{aligned}\right\}l^{(6)}\\
&\frac{1}{6561}\left\{324a^3-810a^2-1458ab+320a+1215b+4374c-1458d-44\right\}l^{(5)}\\
&\qquad+\frac{1}{729}\left\{108a^2-60a-162b-243d+8\right\}l^{(4)}\\
&+\frac{5-18a}{81}l^{(3)}+\frac{3a-1}{9}l^{(2)}-l^{(1)}\\
&+\frac{1}{729}\left\{216a^2-144a-324b+20\right\}p\\
&+\frac{4}{59049}\left\{\begin{aligned}&1701a^4-5676a^3-878a^2b+3780a^2+11664ab+6561b^2+13122ac-17496ad\\&-924a-2268b-10935c+4860d+13122g+77\end{aligned}\right\}q_3\\
&+\frac{1}{729}\left\{-432a^2+240a+648b-972d-32\right\}q_2+\frac{1}{27}(4-12a)q_1\\
&+\frac{4}{19683}\left\{-324a^3+810a^2+1458ab-360a-1215b-4374c+1458d+44\right\}r_3\\
&\qquad+\frac{1}{243}\left\{72a-20\right\}r_2+\frac{4}{3}r_1
\end{aligned}\right\}=(m,n)_3$$

Erit ergo:

$$l^{(6)}=-\frac{4}{3}q_3,\quad l^{(5)}=\frac{2-6a}{27}q_3-\frac{2}{3}r_3-(m,n)_{-3}$$

$$l^{(4)}=\frac{12a-2}{81}q_3-\frac{q_2}{3}-\frac{r_3}{9}-\frac{1}{3}(m,n)_{-3}+(m,n)_{-2}$$

$$\begin{aligned}l^{(3)}=\frac{1}{2187}\left\{-324a^2+180a+810b+405d-24\right\}q_3+\frac{2-6a}{27}q_2+\frac{45a-12}{243}r_3\\+\frac{r_2}{3}-\frac{p}{9}+\frac{9a-2}{81}(m,n)_{-3}+\frac{3a-1}{9}(m,n)_{-2}-(m,n)_0\end{aligned}$$

etc.

Videmus, quantitates $l^{(\alpha)}$ eodem frui denominatore, quo p, q, r

Evolutiones pag. 77 suppeditant valores expressionum $(m,n)\alpha$; sunt autem hae:

$$(3,3)_{-3}=-\frac{1}{3},\quad(3,3)_{-2}=-\frac{2}{9},\ (3,3)_0=\frac{-72a+20}{243},\ (3,3)_1=\frac{135a^2-180a-162b+35}{729},$$

$$(3,3)_3=\frac{-2574a^3-5427a^2-7290ab+2682ab+6840b+13122c-5832d-316}{19683}$$

$$(2,3)_0=-\frac{1}{2},\ (2,3)_1=-\frac{2}{9},\quad(2,3)_3=\frac{-72a+20}{243},\quad(1,3)_3=-\frac{1}{3}$$

$$(3,2)_0 = \frac{2}{9},\ (3,2)_1 = \frac{-6a+4}{27},\ (3,2)_3 = \frac{-108a^2+180a+162b-28}{729}$$

$$(2,2)_3 = \frac{2}{9}$$

$$(3,1)_1 = \frac{1}{3},\qquad (3,1)_3 = \frac{6a-5}{27}$$

$$(3,0)_3 = -\frac{2}{3}$$

omnes aliae ex pressiones $(m,n)_\alpha$, quae in aequationibus nostris inveniuntur, sunt $= O$.

Inventis valoribus quantitatum $l^{(\alpha)}_{m,n,}$ jam facile functio:

$\frac{\phi(xy)_{m,n}}{G(xy)_2}$ ad functiones $H(xy)_\alpha$ et $H'(xy)_\alpha$ revocatur. Est enim:

$$\frac{\phi(xy)_{m,n}}{G(xy)_2} = \frac{l^{(6)}_{m,n}x^2y + l^{(5)}_{m,n}x^2 + l^{(4)}_{m,n}xy + l^{(3)}_{m,n}x + l^{(2)}_{m,n}y + l^{(1)}_{m,n}}{G(xy)_2}$$

$$\text{vel}\quad \frac{\phi(xy)_{m,n}}{G(xy)_2} = l^{(6)}_{m,n}\left\{\frac{1}{4}H'(xy)_1 + \frac{1-3a}{12}H'(xy)_2 + \frac{4a-1}{12}H'(xy)_3 + \frac{2b-3d}{12}H(xy)_3 + \frac{d}{12}H(xy)_2\right\}$$

$$+ l^{(5)}_{m,n}\left\{\frac{1}{2}H'(xy)_2 - \frac{1}{2}H'(xy)_3\right\} + l^{(4)}_{m,n}H'(xy)_3 + l^{(3)}_{m,n}H(xy)_3 + l^{(2)}_{m,n}H(xy)_2 + l^{(1)}_{m,n}H(xy)_1$$

$$= \left\{\frac{1}{4}l^{(6)}_{m,n}\right\}H'(xy)_1 + \left\{\frac{1-3a}{12}l^{(6)}_{m,n} + \frac{1}{2}l^{(2)}_{m,n}\right\}H'(xy)_2 + \left\{\frac{4a-1}{12}l^{(6)}_{m,n} - \frac{1}{12}l^{(5)}_{m,n} + l^{(4)}_{m,n}\right\}H'(xy)_3$$

$$+ l^{(1)}_{m,n}H(xy)_1 + \left\{\frac{d}{12}l^{(6)}_{m,n} + l^{(2)}_{m,n}\right\}H(xy)_2 + \left\{\frac{2b-3d}{12}l^{(6)}_{m,n} + l^{(3)}_{m,n}\right\}H(xy)_3$$

Scribamus brevitatis causa:

$$\frac{\phi(xy)_{m,n}}{G(xy)_2} = h^{(1)}_{m,n}H(xy)_1 + h^{(2)}_{m,n}H(xy)_2 + h^{(3)}_{m,n}H(xy)_3 + h'^{(1)}_{m,n}H'(xy)_1 + h'^{(2)}_{m,n}H'(xy)_2 + h'^{(3)}_{m,n}H'(xy)_3$$

$$= \sum_{1}^{3}{}_{\gamma}\left\{h^{(\gamma)}_{m,n}H(xy)_\alpha + h'^{(\gamma)}_{m,n}H'(xy)_\alpha\right\}$$

Habuimus aequationem 16)

$$F(xy)_{m,n} = \frac{x^m y^{n-1} (n G(xy)_2 - y G(xy)_{2\,2})}{G^3(xy)_2} = \frac{\phi(xy)_{m,n}}{G(xy)_2} + \frac{d}{dx} \frac{\psi(xy)_{m,n}}{G(xy)_2}$$

et 9)

$$\delta Hxy)_{\alpha\beta} = \delta \frac{x^\alpha y^\beta}{G(xy)_2} = -\sum_{\alpha'\beta'} \delta g_{\alpha'\beta'} \frac{x^{\alpha+\alpha'} y^{\beta+\beta'-1} ((\beta+\beta') G(xy)_2 - y G(xy)_{2,2})}{G^3(xy)_2}$$

$$= -\sum_{\alpha'\beta'} \delta g_{\alpha'\beta'} F(xy)_{\alpha+\alpha',\ \beta+\beta'}$$

$$\begin{pmatrix} \alpha' = 0, 1 \\ \beta' = 0, 1, 2 \end{pmatrix}$$

$$\delta \int \overline{H(xy)_{\alpha\beta}\, dx} = -\sum_{\alpha'\beta'} \delta g_{\alpha,\beta'} \int \overline{F(xy)_{\alpha+\alpha',\ \beta+\beta'}\, dx}$$

$$= -\sum_{\alpha',\beta'} \delta g_{\alpha',\beta'} \int \overline{\frac{\phi(xy)_{\alpha+\alpha',\beta+\beta'}}{G(xy)_2}\, dx}$$

$$= -\sum_{\alpha',\beta'} \delta g_{\alpha',\beta'} \sum_{1}^{3}{}_{\gamma} \int \overline{\left\{ h^{(\gamma)}_{\alpha+\alpha'\ \beta+\beta'} H(xy)_\gamma + h'^{(\gamma)}_{\alpha+\alpha',\ \beta+\beta'} H'(xy)_\gamma \right\} dx}$$

$$= -\sum_{1}^{3}{}_{\gamma} \int \overline{H(xy)_\gamma\, dx} \sum_{\alpha',\beta'} \delta g_{\alpha'\beta'} h^{(\gamma)}_{\alpha+\alpha',\ \beta+\beta'}$$

$$- \sum_{1}^{3}{}_{\gamma} \int \overline{H'(xy)_\gamma\, dx} \sum_{\alpha'\beta'} \delta g_{\alpha',\beta'} h'^{(\gamma)}_{\alpha+\alpha',\beta+\beta'}$$

vel etiam

16) $$\delta J_{\alpha\beta} = \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h^{(\gamma)}_{\alpha+\alpha'\ \beta+\beta'} \right\} J_\gamma + \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h'^{(\gamma)}_{\alpha+\alpha'\beta+\beta'} \right\} J'_\gamma$$

Ex integralibus $J_{\alpha,\beta}$ componuntur integralia nostra J_α Fiet:

$$\delta J_1 = \delta \int \overline{\frac{dx}{G(xy)_2}} = \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha'\beta'} - \delta g_{\alpha',\beta'} h^{(\gamma)}_{\alpha',\beta'} \right\} J_\gamma + \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h'^{(\gamma)}_{\alpha',\beta'} \right\} J'_{(\gamma)}$$

$$\delta J_2 = \delta \int \overline{\frac{y\,dx}{G(xy)_2}} = \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h^{(\gamma)}_{\alpha',1+\beta'} \right\} J_\gamma + \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h'^{(\gamma)}_{\alpha',1+\beta'} \right\} J'_\gamma$$

$$\delta J_3 = \delta \int \overline{\frac{y\,dx}{G(xy)_2}} = \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h^{(\gamma)}_{1+\alpha',\beta'} \right\} J_\gamma + \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h'^{(\gamma)}_{1+\alpha',\beta'} \right\} J'_\gamma$$

$$\delta J'_3 = \delta \int \overline{\frac{xy\,dx}{G(xy)_2}} = \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h^{(\gamma)}_{1+\alpha',1+\beta'} \right\} J_\gamma + \sum_{1}^{3}{}_{\gamma} \left\{ \sum_{\alpha',\beta'} - \delta g_{\alpha',\beta'} h'^{(\gamma)}_{1+\alpha'1+\beta'} \right\} J'_\gamma$$

$$\delta J'_2 = \delta \int \overline{\frac{2x^2 + xy}{G(xy)_2}}\, dx = \sum_{1}^{3}{}_{\gamma} \left\{ \sum\nolimits_{\alpha',\beta'} - \delta g_{\alpha',\beta'} \left(2\, h^{(\gamma)}_{2+\alpha',\beta'} + h^{(\gamma)}_{1+\alpha',1+\beta'} \right) \right\} J_\gamma$$

$$+ \sum_{1}^{3}{}_{\gamma} \left\{ \sum\nolimits_{\alpha',\beta'} - \delta g_{\alpha',\beta'} \left(2 h'^{(\gamma)}_{2+\alpha',\beta} + h'^{(\gamma)}_{1+\alpha',1+\beta'} \right) \right\} J'_\gamma$$

$$\delta J'_1 = \delta \int \overline{\frac{dx}{G(xy)_2}} \left\{ 4x^2 y + \frac{6a-2}{3} x^2 - \frac{a}{3} xy + \frac{3d-2b}{3} x - \frac{d}{3} y \right\}$$

$$= 2\,\delta a \left\{ \frac{1}{2} J'_2 - \frac{1}{2} J'_3 \right\} - \frac{\delta a}{3} J'^3 + \left\{ \delta d - \frac{2}{3} \delta b \right\} J_3 - \frac{\delta d}{3} J_2$$

$$+ 4\delta \int \overline{\frac{x^2 y\, dx}{G(xy)_2}} + \frac{6a-2}{3} \delta \int \overline{\frac{x^2\, dx}{G(xy)_2}}$$

$$- \frac{a}{3} \delta \int \overline{\frac{xy\, dx}{G(xy)_2}} + \frac{3d-2b}{3} \delta \int \overline{\frac{x\, dx}{G(xy)_2}} - \frac{d}{3} \delta \int \overline{\frac{y\, dx}{G(xy)_2}}$$

$$= -\frac{4}{3} \delta a J'_3 + \delta a\, J'_2 + \left(\delta d - \frac{2}{3} \delta b \right) J_3 - \frac{\delta d}{3} J_2$$

$$+ \sum_{1}^{3}{}_{\gamma} \left\{ \sum\nolimits_{\alpha',\beta'} - \delta g_{\alpha',\beta'} \left\{ 4\, h^{(\gamma)}_{2+\alpha',1+\beta'} + \frac{6a-2}{3} h^{(\gamma)}_{2+\alpha',\beta'} - \frac{a}{3} h^{(\gamma)}_{1+\alpha' 1+\beta'} + \frac{3d-2b}{3} h^{(\gamma)}_{1+\alpha',\beta'} - \frac{d}{3} h_{\alpha';1+\beta'} \right\} \right\} J_\gamma$$

$$+ \sum_{1}^{3}{}_{\gamma} \left\{ \sum\nolimits_{\alpha',\beta'} - \delta g_{\alpha',\beta'} \left\{ 4 h'^{(\gamma)}_{2+\alpha',1+\beta'} + \frac{6a-2}{3} h'^{(\gamma)}_{2+\alpha',\beta'} - \frac{a}{3} h'^{(\gamma)}_{1+\alpha',1+\beta'} + \frac{3d-2b}{3} h'^{(\gamma)}_{1+\alpha',\beta'} - \frac{d}{3} h_{\alpha',1+\beta'} \right\} \right\} J'_\gamma$$

Hoc modo aequationes differentiales ad periodos integralium abelianorum pro $\rho = 3$ definiendas, formandae mihi esse videntur. Non putaverim autem coëfficientes ipsarum, donec moduli aequationis nostrae $G(xy) = O$ generales maneant, explicite per hosce modulos calculo subducto exprimi posse. Videmus enim illud aequationum linearium systema, quod nobis valores quaesitos coëfficientium suppeditaturum esset, magis esse complicatum, quam ut sperare possemus, fore, ut formulas aliquatenus perspicuas nanciscamur.

VITA.

Natus sum Ludovicus Milewski Posnaniae die VIII mensis Augusti a. h. s. LI, patre Witoldo, matre Josepha e gente a Sarnowska, quibus adhuc superstitibus gaudeo. Fidem profiteor catholicam. Anno h. s. LX gymnasium ad St. Mariam Magdalenam adii, quod patria in urbe tum auspiciis Brettneri, viri doctissimi florebat. Maturitatis testimonio munitus a. LXIX Vratislaviam me contuli, ubi per tria semestria audivi viros Illustrissimos: Bachmann, Braniss, Elvenich, Freymond, Galle, Marbach, O. E. Meyer, Nehring, Rosanes, Schröter. Tum numero civium Universitatis Fridericae Guilelmae a Rectore Magnifico Bruns adscriptus per octo semestria interfui scholis VV. Ill. Dove, du Bois-Reymond, Dühring, Förster, Helmholtz Kronecker, Kummer, Pochhammer, Schneider, Thomé, Weierstrass, Werder, Zeller.

Seminarii mathematici exercitationibus Vratislaviae per unum, Berolini per quinque adfui semestria.

Omnibus viris, optime de me meritis, quorum egregia disciplina atque institutione sum usus, maxime vero Ill. **Kummer** et **Weierstrass,** summas, quas possum, gratias ago semperque habebo.

THESES.

I. Aequationibus differentialibus periodi functionum abelianarum analytice definiuntur. Attamen ad valores ipsarum computandos aliae methodi sunt quaerendae.

II. In ellipticarum functionum theoria non ab integralibus sed a functionibus ipsis per praecipuas quasdam proprietates definiendis proficiscendum esse puto.

III. Calorem non esse materiam, jam inde concludi potest, quod aëres et vapores, quum in vacuum expanduntur, non fiunt frigidiores.

Corrigenda.

Pag.	v.	leg.	loco.
3	3	$x'^{2\rho-\alpha-1}$	$x'^{2\rho-\alpha-2}$
4	22	$4\,(x x')^{2\rho-1}$	$4\,(x x')^{2\rho-2}$
”	23	$-\frac{1}{4}\sum$	$\frac{1}{4}\sum$
22	16	$\phi\,(x)_r$	$\phi\,(x)_\rho$
25	13	$(\rho+s-\mathfrak{a}+1)$	$(s-\mathfrak{a}+1)$
28	10	$\partial\,J'_\beta$	$\partial\,J_\beta$
32	15	$H'_2\ H'_3$	$H_2\ H_3$
36	9	g_{3-m+n}	$\mathfrak{g}_{3-m+n}$
40	16	$+d\,(\,.\,.\,)$	$-d\,(\,.\,.\,)$
44	1	$-\frac{2e_\mu^{\alpha-1}}{R'e_\mu}$	$-\frac{2\,e_\mu^{\alpha-1}}{\sqrt{R'e_\mu}}$
52	10	$\mathfrak{h}'_{s,t}$	$\mathfrak{h}_{s,t}$
”	12	Est autem $\mathfrak{g}'_{4,3}=-1$	Fst autem $\mathfrak{g}'_{4,3}=1$
53	7	$\mathfrak{h}_{5,3}$	$\mathfrak{h}'_{5,3}$
56	16	$\frac{1}{4\,(g_2^3-27g_3^2)}$	$\frac{1}{8\,(g_2^3-27\,g_3^2)}$
72	13	x^m	x^n
73	5	$-$	$=-$
	10	x	y
79	15	6	a
82	32	r_2	r_5
”	33	r_1	r_2
83	24	p, q	$p_1\ q_1$

BEROLINI
TYPIS EXPRESSIT P. STANKIEWICZ
BEUTH-STRASSE 5.